Original: 3/14/2013
Revised: 8/22/2013

TWO STOCHASTIC PORTFOLIO THEORY MODELS

OF A

LOW BETA ANOMALY

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ABSTRACT

The finance literature has established that portfolios of low beta stocks have higher growth rates (long-term return rates) than portfolios of high beta stocks The literature largely concludes from this that low beta stocks have higher growth rates than high beta stocks and that investors are irrational. This paper presents two Stochastic Portfolio Theory models of a Low Beta Anomaly that do not require irrational investor behavior and do not require that low beta stocks have higher growth rates than high beta stocks. Both models imply a Low Beta Anomaly that is due to reconstitution relative volatility capture. One of the models implies outperformance of portfolios of low beta stocks for all benchmarks used to define stocks' betas. The other implies outperformance or underperformance of portfolios of low beta stocks according to whether the benchmark is such that the equal weight portfolio's beta is less than or greater than 1.

Introduction

The quandary

The finance literature has established that portfolios of low beta stocks have higher growth rates (long-term return rates) than portfolios of high beta stocks The literature largely concludes from this that low beta stocks have higher growth rates than high beta stocks.

According to Modern Portfolio Theory (MPT) and the Capital Asset Pricing Model (CAPM), rational investor behavior implies that additional risk is rewarded with additional return. The outperformance of portfolios of low beta stocks has been proclaimed a "Low Beta Anomaly" (LBA). The CAPM has been proclaimed "dead".

Lacking a rational explanation for a LBA, the literature often has turned to explanations based on irrational investor behavior. For example, Behavioral Finance has come to the rescue with several hypotheses that purport to explain a LBA. These include:

- The "Leverage Aversion Hypothesis".

  The Leverage Aversion Hypothesis presumes that investors who seek higher returns by taking higher risk are forced to do so by overweighting higher beta stocks, because they cannot lever lower beta stocks. This is said to result in overvaluing high beta stocks relative to low beta stocks, hence producing a growth rate advantage for low beta stocks.

- The "Preference For Gambling Hypothesis".

  The Preference For Gambling Hypothesis presumes that investors' utility functions do not have monotonically declining slopes everywhere, hence have regions of risk loving instead of risk aversion. This is said to result in overvaluing high beta stocks relative to low volatility stocks, hence producing a growth rate advantage for low beta stocks.

- The "Excessive Optimism Hypothesis".

  The Excessive Optimism Hypothesis presumes that investors tend to be too optimistic concerning growth stocks, which tend to be high beta stocks. This is said to result in overvaluing growth stocks relative to non-growth stocks, hence producing a growth rate advantage for low beta, non-growth stocks.

- The "Delegated-Agency Hypothesis".

  The Delegated-Agency Hypothesis presumes that investors are benchmark centric. If so, the argument goes, investors consider low beta stocks as contributing excessively to tracking error, hence underweighting them. This is said to result in overvaluing high beta stocks relative to low beta stocks, hence producing a growth rate advantage for low beta stocks.

Explanations that depend on low beta stocks having higher growth rates than high beta stocks can be problematic. Some Finance literature explanations imply market behavior that is inconsistent with observed market behavior, and fail to distinguish between portfolios' growth rates and their stocks' growth rates. There are other explanations for a LBA that do not require irrational investor behavior, and do not require that low beta stocks have higher growth rates than high beta stocks.

If one presumes that low beta stocks remain low beta stocks, then, if they have higher growth rates than high beta stocks, low beta stocks would become the market's larger stocks and high beta stocks would become the market's smaller stocks. The market would become increasingly concentrated in the largest (low beta) stocks. The market does not exhibit this behavior. Instead, the market's capital distribution across size appears to be mean reverting to a reasonable shape (Fernholz, 2002). Thus, any explanation for a LBA that presumes that low beta stocks have higher growth rates than high beta stocks and that low beta stocks remain low beta stocks is problematic.

Suppose low beta stocks have the same growth rates as high beta stocks and low beta stocks remain low beta stocks. Reasonable assumptions about stocks' randomness (stochastic disturbances around trend) imply that the market will still spend most of the time concentrated in a few of the largest stocks (which will tend to be high beta stocks) (Fernholz, 2002).[1] Thus, any explanation for a LBA anomaly that presumes that low beta stocks have the same growth rates as high beta stocks may be problematic.

One way out of this quandary is to drop the constant growth rate assumption and to presume that low beta stocks transition to high beta stocks, and vice versa. Then, there are plausible models for a LBA based on Fernholz's Stochastic Portfolio Theory (Fernholz 1982, 2002).

Since stocks' beta source volatility is likely to be a substantial portion of their total volatility, models of a LBA also may be the basis of a Low volatility anomaly (LVA).

Stochastic Portfolio Theory (SPT)

On the surface, it appears that if low beta stocks have the same growth rates as high beta stocks, portfolios of low beta stocks would have the same growth rates as portfolios of high beta stocks and there would be no LBA. This is not so. SPT shows that a portfolio's growth rate is not the weighted average of its stocks' growth rates and that long-only portfolios' growth rates exceed the weighted average of their stocks' growth rates. Thus, a presumption that the LBA is due to low beta stocks having higher growth rates than high

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[1] High beta stocks tend to be high residual volatility stocks. High residual volatility stocks' prices diverge faster than low residual volatility stocks' prices. Over time, both tails of stocks' price distribution become dominated by high residual volatility stocks. These will tend to be high beta stocks to the extent that beta and residual volatility are positively correlated.

beta stocks is unwarranted. Several plausible SPT based models for a LBA are portfolio effects with no corresponding stock effects.[2]

SPT shows that the growth rate of a portfolio that is rebalanced to target weights which are functions of time is the sum of its stocks' weighted average growth rates plus an Excess Growth Rate (EGR). The portfolio's EGR is one-half the difference between the weighted average variance rate of its stocks and the portfolio's variance rate. Equivalently, the portfolio's EGR is one half the difference between the weighted average relative variance rate of its stocks and the portfolio's relative variance rate, where the stocks' and the portfolio's returns are measured relative to a benchmark.[3] SPT also shows that a long-only portfolio's EGR is positive. A long-only portfolio of this type has a growth rate that exceeds the weighted average growth rates of its stocks.

The EGR of this type of portfolio can be thought of as arising from the trading necessary to maintain its target weights. Consider an equal-weight portfolio (EWP), as an example. Positions in stocks that outperform the EWP must be reduced and positions in stocks that underperform the EWP must be increased to retain the EWP's equal weights. This trading has a buy after a decline and sell after a rise character, and a buy low – sell high character as stocks cycle up and down relative to the EWP.[4] Since the resulting trading profit depends on stocks' volatilities relative to the portfolio, it is termed "relative volatility capture".

A specific equal weight portfolio is an understandable way of illustrating a portfolio's EGR. Suppose the equal weight portfolio contains a large number of identical stocks, each with a 50/50 chance of going up 100% or down 50% each period. Assume that the stocks' returns are independent across stocks and time. Each stock's expected arithmetic return is (0.5*100-0.5*50) = 25%. However, each stock's growth rate (long-term return rate) is only 0%. To see this, consider what happens over a large number of periods. The stock will go up 100% in about 50% of the periods and will go down 50% in about 50% of the periods (for the same reason that flipping a coin yields about 50% heads and 50% tails in a large number of flips). A typical two period result contains one up 100% and one down 50%. Up 100% represents multiplying the initial price by 2 and down 50% represents multiplying the initial price by 0.5. No matter which comes first, the net result is to multiply the stock's initial price by 2*0.5=0.5*2 = 1 over a typical two period interval. This corresponds to a two period growth rate of 0%, or a growth rate (long term return rate) of 0% per period.

Now consider what happens to the equal weight portfolio. Each period, about half of its stocks go up 100% and about half of its stocks go down 50%. Diversification results in the

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2 Some of them can produce a LBA even when low beta stocks have lower, growth rates than high beta stocks.

3 The benchmark is arbitrary.

4 This trading has the same trading character of a short synthetic call, which requires buying the stock after it declines and selling the stock after it rises. That trading results in earning the call premium in the form of a trading profit.

portfolio's return being about 25%, every period. The portfolio's growth rate is about 25% per period, even though its stocks' growth rates are 0% per period. The EGR formula for this portfolio computes to 25% per period.

The equal weight portfolio must be rebalanced each period to gain in this fashion. This requires increasing the positions of stocks that go down 50% and decreasing the positions of stocks that go up 100%. This trading has a buy low – sell high character as stocks cycle up and down relative to the equal weight portfolio. Thus, the trading required to rebalance the equal weight portfolio generates a trading profit that is equal to the portfolio's EGR, and the EGR represents the portfolio's relative volatility capture.

Fernholz (2002) provides the mathematics implying that equal weight portfolios should outperform their capitalization weight counterparts due to relative volatility capture, as opposed to higher growth rates for small stocks vs. large stocks. Greene and Rakowski (2011) provides the empirical evidence that this is so.

SPT implies that the growth rate of a portfolio that is rebalanced to target weights (which may vary over time) and is reconstituted according to stocks' size rank or other classification schemes, e.g., portfolios of small stocks or low beta stocks, is the sum of its stocks' weighted average growth rates plus its EGR, plus additional terms that account for reconstitution relative volatility capture.[5]

Understanding reconstitution relative volatility capture is facilitated by considering the SPT explanation of the traditional size effect, as exemplified by the well known outperformance of small stock portfolios with respect to large stock portfolios. The SPT explanation presented here depends on the concept of market stability, which has to do with the market's capital distribution.[6]

The market's capital distribution curve (CDC) is a plot of stocks' market weights (vertical axis) against their size rank from large to small (horizontal axis). The market is said to be stable if its CDC is mean reverting to a typical shape that spreads capital reasonably across its stocks. An unstable market is characterized by extreme concentration in one or a few of the largest stocks most of the time. In an unstable market, the CDC typically has a market weight close to 1 at size rank 1 or, perhaps, a cumulative market weight of close to 1 spread across a few more size ranks, and market weights close to 0 for all other size ranks. Fernholz's (2002) empirical analysis shows that large stock universes in developed countries are stable.

Suppose that the market is stable. For illustrative purposes, assume that its capital distribution curve is fixed. Then at any point in time, the aggregate market weight of the largest

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[5] Fernholz distinguishes between portfolio weights determined by stocks' size rank and portfolio weights associated with reconstitutions involving stocks that enter or leave the portfolio. This paper denotes all non EGR source relative volatility capture as "reconstitution relative volatility capture".

[6] Equal weight portfolios outperform their capitalization weighted counterparts in some unstable markets, too. However, the stable market case is simpler to understand and has empirical support.

100 stocks' is the same as at any other point in time. Consider a capitalization weighted portfolio of the market's largest 100 stocks at time 0. Suppose its capitalization is 15% of the market's total capitalization. At a later time, time 1, the then largest 100 stocks also have a capitalization of 15% of the market's. However, due to randomness, some of the time 0 largest 100 stocks will be smaller than all the time 1 largest 100 stocks. Thus, the time 0 portfolio represents less than 15% of the time 1 market capitalization, hence has underperformed the market.

At time 1, the portfolio of the largest 100 stocks must be reconstituted by selling those of its stocks that are no longer among the largest 100 stocks and buying those stocks which have replaced them in the largest 100 category. In relative return terms, stocks must be sold after a decline and bought after a rise. As stocks cycle in and out of the largest 100 stocks category, this trading has a buy high – sell low character that leads to the large stock portfolio's underperformance. This negative trading profit is negative reconstitution relative volatility capture.

In this market, the aggregate market weight of the smallest 100 stocks' also is the same at any point in time. Consider a capitalization weighted portfolio of the market's smallest 100 stocks at time 0. Suppose its capitalization is 1% of the market's total capitalization. At a later time, time 1, the then smallest 100 stocks also have a capitalization of 1% of the market's. However, due to randomness, some of the time 0 smallest 100 stocks are larger than all the time 1 smallest 100 stocks. Thus, the time 0 portfolio represents more than 1% of the time 1 market capitalization, hence has outperformed the market.

At time 1, the portfolio of the smallest 100 stocks must be reconstituted by selling those of its stocks that are no longer one of the smallest 100 stocks and buying those stocks which have replaced them in the smallest 100 category. In relative return terms, stocks must be bought after a decline and sold after a rise. As stocks cycle in and out of the smallest 100 stocks category, this trading has a buy low – sell high character, that leads to the small stock portfolio's outperformance. This positive trading profit is positive reconstitution relative volatility capture.

The large and small stock portfolios' performance stems from their trading, which stems from stocks' relative volatility.

The same argument and conclusion holds for portfolios of the largest and smallest stocks no matter what boundaries are used, e.g., for portfolios of the largest half of the stocks and the smallest half of the stocks, or portfolios of the largest X% of the stocks and the smallest X% of the stocks or portfolios of the largest X% of the stocks and the smallest (1-X)% of the stocks.

Fernholz (2002) provides a mathematical proof that portfolios of small stocks should outperform portfolios of large stocks in a stable market.

SPT shows that a stable market of stocks with differential growth rates requires, in some sense, that the market's smaller stocks have higher growth rates than its larger stocks. To see this, consider a simple market of stocks with the same growth rates, the same standard deviations, and assume that all correlations across stocks and time are zero. Assume that stocks' continuous returns are normally distributed. This implies that stocks' cumulative continuous relative returns are equivalent to a sample from a single normal distribution with zero mean and standard deviation proportional to the square root of time.

This apparently innocuous market is unstable, and tends to spend most of its time concentrated in one or a few stocks. This is due to the cumulative random disturbances of its stocks driving the stocks' prices and benchmark weights apart. If it is assumed that stocks' growth rates are inversely related to their market weights, e.g., proportional to equal-weight less actual weight, then the market's smaller stocks will have higher growth rates than the market's larger stocks. These differential growth rates will tend to drive the stocks' benchmark weights together and the market toward equal weight. The result is a balance between the divergence effect of stocks' randomness and the convergence effect of their differential growth rates. The market will be stable, with a capital distribution across size rank that is mean reverting to a reasonable shape.[7]

Fernholz (2002) examines about a 5,000 stock universe and shows that, for about the largest 1,000 stocks, the smaller stocks tend to have lower estimated growth rates than the larger stocks, contrary to the conclusions drawn in the literature based on portfolios of small stocks vs. portfolios of large stocks.[8] For the remaining about 4,000 stocks, the tendency is for the smaller stocks to have higher growth rates than the larger stocks. However, only about the universe's smallest 1,500 have estimated growth rates that exceed those of the universe's largest stocks. In this universe, stability appears to be achieved by higher growth rates of stocks smaller than those included in many popular large stock benchmarks.

SPT applied to the low beta anomaly

Portfolios' relative volatility capture through their EGRs and reconstitutions occurs in the same time frame as their trading. Thus the relevant covariance matrix for computing stocks' betas for the models presented here has the reconstitution interval time frame.[9]

Stocks' betas are measured with respect to a benchmark. In what follows, it is assumed that the benchmark is the stock universe used to establish the existence of the LBA.

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7 Fernholz (2002) shows that stability can be achieved by assigning the same growth rate to all the stocks except for the smallest, and assigning a slightly higher growth rate to the smallest stock. He terms this the "Atlas" model. He interprets the observed market as all but the smallest stock. In this model, the observed market's stocks have the same growth rate, the observed market is stable, and there are births and deaths in the observed market.

8 This does not imply an inconsistency with the empirics in the literature. It only implies that the small stock portfolio vs. large stock portfolio performance differentials are more likely due to portfolio effects, e.g., reconstitution relative volatility capture, than stocks' differential growth rates.

9 The models are formulated in continuous time, which corresponds to point in time covariance matrices. Stocks' covariance matrix is time dependent and time frame dependent, and stocks' betas depend on both stocks' covariance matrix and the benchmark's weights. Thus, the betas discussed in this paper are stocks' actual betas, and do not necessarily correspond closely to the estimated betas used in the LBA literature.

Understanding why changes in stocks' betas can cause a LBA requires an understanding of what are stocks' betas and why and how their betas can change in ways that cause a LBA.

Stocks' betas depend on their benchmark weights

A stock's beta is its covariance with the benchmark divided by the benchmark's variance. Both the numerator and denominator of this ratio are functions of the benchmark's weights. However, it is only the stocks' covariances with the benchmark that are unique to each stock, since the benchmark's variance is common to all of the stocks.

Suppose stocks' covariance matrix does not change. Then stocks' betas depend only on their benchmark weights. If a stock's benchmark weight changes, so will its beta. Betas that change systematically with benchmark weights can produce a LBA.

Betas based on common variances and common correlations and how they can cause a LBA

The covariance matrix used in this section is unrealistic. Nevertheless, the phenomenon it illustrates may, at times, be present in realistic covariances matrices; hence the consequent beta behavior may be relevant to an observed LBA.

Assume that stocks have identical variances and identical positive correlations. Then all stocks have the same growth rate, and there are no differential growth rates or differential long term returns.

In this model, all stocks have the same variances, hence there are no low volatility stocks versus high volatility stocks and no possibility of a stock level LVA.

These stocks' betas are:

$$\beta_{i\mu} = \left[\rho + (1-\rho)\mu_i\right]\left(\frac{\sigma^2}{\sigma_\mu^2}\right) \tag{1}$$

$\beta_{i\mu} \equiv$	Stock i's beta with respect to the benchmark.
$\rho \equiv$	Stocks' common correlation with each other.
$\mu_i \equiv$	Stock i's benchmark weight.
$\sigma^2 \equiv$	Stocks' common variance.
$\sigma_\mu^2 \equiv$	The benchmark's variance.

Equation (1) shows that stocks' betas are a monotonically increasing function of their benchmark weights. Larger stocks will have higher betas than smaller stocks. If one stock is larger than another, it will have a higher beta than the other. Ranking stocks by beta is the same as ranking stocks by size.

Consider a capitalization weighted portfolio of the benchmark's low beta stocks and a capitalization weighted portfolio of the benchmark's high beta stocks, each formed based on stocks' betas at time 0. The low beta portfolio is allocated 100/0 to the low and high beta stocks, respectively, at time 0. The high beta portfolio is allocated 0/100 to the low and high beta stocks, respectively, at time 0.

These low beta and high beta portfolios are equivalent to the time 0 small stock and large stock portfolios discussed earlier, hence the low beta portfolio will outperform the high beta portfolio for the same reason that the small stock portfolio outperformed the large stock portfolio. This LBA is due to reconstitution relative volatility capture.

At time 1, the low beta portfolio must be reconstituted by selling those of its stocks that are no longer low beta and buying time 0 high beta stocks that have become time 1 low beta stocks. In this model, this amounts to buying time 0 large stocks that have become small and selling time 0 small stocks that have become large. In relative return terms, stocks must be bought after a decline and sold after a rise. As stocks cycle in and out of the low beta (small) category, this trading has a buy low – sell high character that leads to the low beta stock portfolio's outperformance. This trading profit is positive reconstitution relative volatility capture.

At time 1, the high beta portfolio must be reconstituted by selling those of its stocks that are no longer high beta and buying time 0 low beta stocks that have become time 1 high beta stocks. In this model, this amounts to buying time 0 small stocks that have become large and selling time 0 large stocks that have become small. In relative return terms, stocks must be bought after a rise and sold after a decline. As stocks cycle in and out of the high beta (large) category, this trading has a buy high – sell low character that leads to the high beta stock portfolio's underperformance. This trading loss is negative reconstitution relative volatility capture.

Since betas are continuous functions of stocks' variances and correlations, the above conclusions hold when stocks' variances are unequal and when stocks' correlations are unequal, but not too different from each other. There is a positive correlation of 1.0 between stocks' betas and size when stocks' differential variances and differential correlations are zero. As the magnitudes of stocks' differential variances and differential correlations increase from zero, the correlation between stocks' betas and size will decline, but must remain positive for a while. As long as it is positive, a low beta portfolio is likely to overweight the benchmark's smaller stocks relative to a high beta portfolio, leading to the same performance advantage for the low beta portfolio as before.

Stocks' total variances are the sum of their beta source variances and their residual variances. Since stocks' beta source volatility is likely to be a substantial portion of their total volatility, a portfolio of low volatility stocks is likely to have trading that is positively correlated with the trading of a portfolio of low beta stocks, hence also is likely to experience positive reconstitution relative volatility capture. If so, there will be a corresponding LVA.

Betas in general and how they can cause a LBA: Model 1

The above described LBA depends on there being a positive correlation between stocks' betas and their benchmark weights. The appendix shows that, for an arbitrary covariance matrix, this correlation is:

$$\rho_{\beta\mu} = \left(\frac{1}{N}\right)\left(\frac{1-\beta_{EW}}{\sigma_\beta \sigma_\mu}\right) \tag{2}$$

$\rho_{\beta\mu} \equiv$	The correlation between stocks' betas and their benchmark weights.
$\beta_{EW} \equiv$	The equal weight portfolio's beta, with respect to the benchmark.
$\sigma_\beta \equiv$	The standard deviation of stocks' betas.
$\sigma_\mu \equiv$	The standard deviation of the benchmark's weights.
$N \equiv$	The number of stocks in the benchmark.

Equation (2) shows that the equal weight portfolio's beta must be less than 1 for the required correlation to be positive and for the consequent LBA to be positive. If the equal weight portfolio's beta exceeds 1, the corresponding LBA is negative. Whether or not this LBA is positive, and for what benchmarks and when, is an empirical issue.

To provide a rough estimate of whether this LBA is likely to be positive or negative, consider a benchmark similar to the Russell 1000 index. For such a benchmark, ballpark values for the equal weight portfolio's beta, the standard deviation of stocks' betas, and the standard deviation of stocks' benchmark weights are about 1.15, 0.50, and 0.0025, respectively. Substitution in Equation (2) yields a correlation between stocks' betas and their benchmark weights of -0.12. This suggests a small negative LBA due to size related reconstitution relative volatility capture. However, the stock betas used in this approximation are regression based beta estimates, which are not the stocks' instantaneous true betas contemplated by the theory. Consequently, a positive LBA due to size related reconstition relative volatility capture should not be ruled out.

Betas in general and how they can cause a LBA: Model 2

Consider a general covariance matrix that does not change and increase stock i's benchmark weight by $d\mu$ and decrease stock j's benchmark weight by $d\mu$. Then, the appendix shows that the differential change in the stocks' betas is:

$$\left(d\beta_i - d\beta_j\right) = \left(\frac{d\mu}{\sigma_\mu^2}\right)\left(\sigma_{i-j}^2 - 2\sigma_\mu^2\left(\beta_i - \beta_j\right)^2\right) \tag{3}$$

$\sigma_{i-j}^2 \equiv$ The variance of a hedge portfolio that is long stock i and short stock j.

Since $\sigma_{i-j}^2 > 0$, $\left(d\beta_i - d\beta_j\right) > 0$ when the stocks' betas are the same. Since $d\mu_i > 0$ and $d\mu_j < 0$, this implies, in continuous time, that if two stocks of adjacent beta ranks interchange their beta rank then the previously lower beta stock has outperformed the previously higher beta stock. This implies that the reconstitution trading for a continuously rebalanced portfolio of low beta stocks will have a buy after a negative relative return – sell after a positive relative return character and that the reconstitution trading for a continuously rebalanced portfolio of high beta stocks will have a buy after a positive relative return and sell after a negative relative return character. There will tend to be positive reconstitution relative volatility capture for the portfolio of low beta stocks and negative reconstitution relative volatility capture for the portfolio of high beta stocks. There will be a positive LBA due to reconstitution relative volatility capture that is unrelated to size.

Since stocks' beta source volatility is likely to be a substantial portion of their total volatility, a portfolio of low volatility stocks is likely to have trading that is positively correlated with the trading of a portfolio of low beta stocks, hence also is likely to experience positive reconstitution relative volatility capture. If so, there will be a corresponding positive LVA that is unrelated to size.

Conclusion

The finance literature has established that portfolios of low beta stocks have higher growth rates (long-term return rates) than portfolios of high beta stocks The literature largely concludes from this that low beta stocks have higher growth rates than high beta stocks and that investors are irrational. This paper presents two Stochastic Portfolio Theory models of a Low Beta Anomaly that do not require irrational investor behavior and do not require that low beta stocks have higher growth rates than high beta stocks. Both models imply a Low Beta Anomaly that is due to reconstitution relative volatility capture. One of the models implies outperformance of portfolios of low beta stocks for all benchmarks used to define stocks' betas. The other implies outperformance or underperformance of portfolios of low beta stocks according to whether the benchmark is such that the equal weight portfolio's beta is less than or greater than 1.

The models presented in the paper attribute a Low Beta Anomaly to portfolios' reconstitutions, in the form of reconstitution relative volatility capture, hence they are a property of portfolios, not stocks.

Since stocks' beta source volatility is likely to be a substantial portion of their total volatility, a portfolio of low volatility stocks is likely to have trading that is positively correlated with the trading of a portfolio of low beta stocks, hence also is likely to experience reconstitution relative volatility capture. If so, there will be a corresponding LVA.

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Appendix

Stocks' and portfolios' betas.

A stock's beta is defined as its covariance with a benchmark divided by the benchmark's variance. This is a point in time definition.

$$\underline{\beta} = \frac{\underline{\underline{\sigma_{1,\mu}}}}{\sigma_{\mu}^{2}} = \frac{\underline{\underline{V_R}}\,\underline{\mu}}{\underline{\mu}'\underline{\underline{V_R}}\,\underline{\mu}} \tag{4}$$

$\underline{\beta} \equiv$	A column vector of stocks' betas.
$\underline{\sigma_{1\mu}} \equiv$	A column vector of stocks' covariances with the benchmark.
$\sigma_{\mu}^{2} \equiv$	The benchmark's variance.
$\underline{\underline{V_R}} \equiv$	Stocks' covariance matrix.
$\underline{\mu} \equiv$	A column vector of the benchmark's stock weights.

A portfolio's beta is the weighted average of its stocks' betas. Denote the column vector of a portfolio's weights by $\underline{\pi}$. Then, Equation (4) implies that the portfolio's beta, β_{π}, is:

$$\beta_{\pi} = \underline{\pi}'\underline{\beta} = \frac{\underline{\pi}'\underline{\underline{V_R}}\,\underline{\mu}}{\underline{\mu}'\underline{\underline{V_R}}\,\underline{\mu}} \tag{5}$$

The correlation between stocks' betas and stocks' benchmark weights.

The correlation between two quantities is their covariance divided by the product of their standard deviations. The following formula for the covariance is used.

$$\sigma_{XY} = \left(\frac{1}{N}\right)\underline{X}'\underline{Y} - \left(\frac{1}{N}\right)\left(\underline{1}'\underline{X}\right)\left(\frac{1}{N}\right)\left(\underline{1}'\underline{Y}\right) \tag{6}$$

The covariance between betas and benchmark weights, $\sigma_{\beta\mu}$, is:

$$\sigma_{\beta\mu} = \left(\frac{1}{N}\right)\underline{\beta}'\underline{\mu} - \left(\frac{1}{N}\right)\left(\underline{1}'\underline{\beta}\right)\left(\frac{1}{N}\right)\left(\underline{1}'\underline{\mu}\right)$$

(7)

$\underline{\beta}'\underline{\mu}$ is the benchmark's beta, hence is 1.0. $\left(\underline{1}'\underline{\mu}\right)$ is the sum of the benchmark's weights, hence is 1. $\left(\frac{1}{N}\right)\underline{1}$ is a column vector of the equal weight portfolio's weights, $\underline{\pi_{EW}}$. Denote the equal weight portfolio's beta by β_{EW}.

$$\sigma_{\beta\mu} = \left(\frac{1}{N}\right)\left(1 - \beta_{EW}\right)$$

(8)

The correlation between stocks' betas and stocks' benchmark weights is:

$$\rho_{\beta\mu} = \left(\frac{1}{N}\right)\left(\frac{1 - \beta_{EW}}{\sigma_\beta \sigma_\mu}\right)$$

(9)

Changes in stocks' betas with respect to stocks' benchmark weights.

The partial derivative of stocks' betas with respect to stocks' benchmark's weights is:

$$\frac{\partial \underline{\beta}}{\partial \underline{\underline{\mu}}} = \frac{\underline{\underline{V_R}}}{\underline{\mu}'\underline{\underline{V_R}}\underline{\mu}} - 2\frac{\left(\underline{\underline{V_R}}\underline{\mu}\right)\left(\underline{\underline{V_R}}\underline{\mu}\right)'}{\left(\underline{\mu}'\underline{\underline{V_R}}\underline{\mu}\right)^2}$$

(10)

Equation (10) can be rewritten as follows.

$$\frac{\partial \underline{\beta}}{\partial \underline{\underline{\mu}}} = \frac{\underline{\underline{V_R}}}{\underline{\mu}'\underline{\underline{V_R}}\underline{\mu}} - 2\underline{\beta}\underline{\beta}'$$

(11)

The small changes in beta produced by small changes in stocks' benchmark's weights can be written as follows.

$$\underline{d\beta} = \left(\frac{\underline{\underline{V_R}}}{\underline{\mu}'\underline{\underline{V_R}}\underline{\mu}} - 2\underline{\beta}\underline{\beta}'\right)\underline{d\mu}$$

(12)

Equation (12) must be interpreted with care. Since the sum of the benchmarks weights satisfies $\underline{1}'\underline{\mu}=1$, it must be true that $\underline{1}'d\underline{\mu}=0$.

Changes in stocks' betas with respect to stocks' covariance matrix.

The small changes in stocks' betas produced by small changes in stocks' covariance matrix are:

$$d\underline{\beta}=\frac{\partial_V\left(\underline{\underline{V_R}}\,\underline{\mu}\right)}{\underline{\mu}'\underline{\underline{V_R}}\,\underline{\mu}}-\frac{\underline{\underline{V_R}}\,\underline{\mu}}{\left(\underline{\mu}'\underline{\underline{V_R}}\,\underline{\mu}\right)^2}\partial_V\left(\underline{\mu}'\underline{\underline{V_R}}\,\underline{\mu}\right) \tag{13}$$

$$d\underline{\beta}=\left(\frac{d\underline{\underline{V_R}}\,\underline{\mu}}{\underline{\mu}'\underline{\underline{V_R}}\,\underline{\mu}}\right)-\underline{\beta}\left(\frac{\underline{\mu}'d\underline{\underline{V_R}}\,\underline{\mu}}{\underline{\mu}'\underline{\underline{V_R}}\,\underline{\mu}}\right) \tag{14}$$

Stocks' total differential beta.

Suppose stocks' benchmark weights change and stocks' covariance matrix changes, both by small amounts. Then stocks' betas change as follows.

$$d\underline{\beta}=\left[\left(\frac{d\underline{\underline{V_R}}\,\underline{\mu}}{\underline{\mu}'\underline{\underline{V_R}}\,\underline{\mu}}\right)-\underline{\beta}\left(\frac{\underline{\mu}'d\underline{\underline{V_R}}\,\underline{\mu}}{\underline{\mu}'\underline{\underline{V_R}}\,\underline{\mu}}\right)\right]+\left(\frac{\underline{\underline{V_R}}}{\underline{\mu}'\underline{\underline{V_R}}\,\underline{\mu}}-2\underline{\beta}\,\underline{\beta}'\right)d\underline{\mu} \tag{15}$$

A LBA because outperforming stocks' betas tend to rise relative to underperforming stocks' betas.

Suppose stocks' covariance matrix does not change.

Consider increasing stock i's benchmark weight by $d\mu$ and decreasing stock j's benchmark weight by $d\mu$. Then Equation (12) provides useful insight.

$$d\underline{\beta}=\left(\left(\frac{1}{\sigma_\mu^2}\right)\underline{\underline{V_R}}\begin{bmatrix}\underline{0}\\ d\mu\\ \underline{0}\\ -d\mu\\ \underline{0}\end{bmatrix}-2\underline{\beta}\,\underline{\beta}'\begin{bmatrix}\underline{0}\\ d\mu\\ \underline{0}\\ -d\mu\\ \underline{0}\end{bmatrix}\right)$$

(16)

$$\underline{d\beta} = \left(\frac{d\mu}{\sigma_\mu^2}\right)\left(\begin{bmatrix}\sigma_{1i}-\sigma_{1j}\\ \vdots \\ \sigma_{ki}-\sigma_{kj}\\ \vdots \\ \sigma_{ni}-\sigma_{nj}\end{bmatrix} - 2\sigma_\mu^2\left(\beta_i-\beta_j\right)\underline{\beta}\right)$$

(17)

Equation (17) implies that stock i's and stock j's beta changes are:

$$d\beta_i = \left(\frac{d\mu}{\sigma_\mu^2}\right)\left(\left(\sigma_{ii}-\sigma_{ij}\right)-2\sigma_\mu^2\left(\beta_i-\beta_j\right)\beta_i\right)$$

(18)

$$d\beta_j = \left(\frac{d\mu}{\sigma_\mu^2}\right)\left(\left(\sigma_{ji}-\sigma_{jj}\right)-2\sigma_\mu^2\left(\beta_i-\beta_j\right)\beta_j\right)$$

(19)

Stock i's differential beta change with respect to stock j is:

$$\left(d\beta_i - d\beta_j\right) = \left(\frac{d\mu}{\sigma_\mu^2}\right)\left(\left(\sigma_{ii}-2\sigma_{ij}+\sigma_{jj}\right)-2\sigma_\mu^2\left(\beta_i-\beta_j\right)\left(\beta_i-\beta_j\right)\right)$$

(20)

$$\left(d\beta_i - d\beta_j\right) = \left(\frac{d\mu}{\sigma_\mu^2}\right)\left(\sigma_{i-j}^2-2\sigma_\mu^2\left(\beta_i-\beta_j\right)^2\right)$$

(21)

$\sigma_{i-j}^2 \equiv \left(\sigma_{ii}-2\sigma_{ij}+\sigma_{jj}\right)$ is the variance of a hedge portfolio that is long stock i and short stock j, hence is nonnegative.

Equation (21) shows that increasing stock i's benchmark weight at the expense of stock j's benchmark weight increases stock i's beta relative to stock j's beta only if:

$$\sigma_{i-j}^2 > 2\sigma_\mu^2\left(\beta_i-\beta_j\right)^2$$

(22)

$$\sigma_{i-j} > \sqrt{2}\sigma_\mu\left|\beta_i-\beta_j\right|$$

(23)

This is assured for stocks whose betas are equal or not too different. Inequalities (22) and (23) are also likely to be satisfied for many other stocks, too.

The constraint placed on stocks by Inequalities (22) and (23) can be quite loose. Consider two stocks that have the same volatilities that are twice the benchmark's volatility, and a differential beta of 1.0.

$$\sigma_{i-j}^2 = \sigma^2 2\left(1-\rho_{ij}\right) > 2\sigma_\mu^2\left(\beta_i - \beta_j\right)^2 \tag{24}$$

$$2\left(1-\rho_{ij}\right) > 1 \tag{25}$$

$$\rho_{ij} < \frac{3}{4} \tag{26}$$

Stocks' correlations are almost always below this required level.

Suppose stock i's beta, and residual standard deviation are 1.2 and 0.3, respectively, and stock j's beta and residual standard deviation are 0.75 and 0.2, respectively. Then stock i's standard deviation is 0.35 and stock j's standard deviation is 0.23. All these values are reasonable, but correspond to larger differences between stocks than average. If the benchmark's standard deviation is taken to be 0.15, also a reasonable level, then the correlation between the two stocks is about 0.25.

It appears that most outperforming stocks' betas are likely to rise relative to most underperforming stocks' betas.