Stochastic Portfolio Theory: An Overview *

ROBERT FERNHOLZ
INTECH
One Palmer Square
Princeton, NJ 08542, USA
bob@enhanced.com

IOANNIS KARATZAS
Departments of Mathematics and Statistics
Columbia University
New York, NY 10027, USA
ik@math.columbia.edu

November 24, 2006

Abstract

Stochastic Portfolio Theory is a flexible framework for analyzing portfolio behavior and equity market structure. This theory was introduced by E.R. Fernholz in the papers (*Journal of Mathematical Economics*, 1999; *Finance & Stochastics*, 2001) and in the monograph *Stochastic Portfolio Theory* (Springer 2002). It was further developed in the papers Fernholz, Karatzas & Kardaras (*Finance & Stochastics*, 2005), Fernholz & Karatzas (*Annals of Finance*, 2005), Banner, Fernholz & Karatzas (*Annals of Applied Probability*, 2005), and Karatzas & Kardaras (2006). It is a descriptive theory, as opposed to a normative one; it is consistent with observable characteristics of actual portfolios and markets; and it provides a theoretical tool which is useful for practical applications.

As a theoretical tool, this framework offers fresh insights into questions of market structure and arbitrage, and can be used to construct portfolios with controlled behavior. As a practical tool, Stochastic Portfolio Theory has been applied to the analysis and optimization of portfolio performance and has been the basis of successful investment strategies for over a decade.

*To appear in *"Mathematical Modelling and Numerical Methods in Finance"* (Alain Bensoussan and Qiang Zhang, Editors), Special Volume of the *"Handbook of Numerical Analysis"*.

Table of Contents

Introduction

Stochastic Portfolio Theory (SPT), as we currently think of it, began in 1995 with the manuscript "On the diversity of equity markets", which eventually appeared as Fernholz (1999) in the *Journal of Mathematical Economics*. Since then SPT has evolved into a flexible framework for analyzing portfolio behavior and equity market structure that has both theoretical and practical applications. As a theoretical methodology, this framework provides insight into questions of market behavior and arbitrage, and can be used to construct portfolios with controlled behavior under quite general conditions. As a practical tool, SPT has been applied to the analysis and optimization of portfolio performance and has been the basis of successful equity investment strategies for over a decade.

SPT is a *descriptive theory*, which studies and attempts to explain observable phenomena that take place in equity markets. This orientation is quite different from that of the well-known *Modern Portfolio Theory of Dynamic Asset Pricing (DAP),* in which market structure is analyzed under strong normative assumptions regarding the behavior of market participants. It has long been suggested that the distinction between descriptive and normative theories separates the natural sciences from the social sciences, and if this dichotomy is valid, then SPT resides with the natural sciences.

SPT descends from the "classical portfolio theory" of Harry Markowitz (1952), as is the rest of mathematical finance. At the same time, it represents a rather significant departure from some important aspects of the current theory of Dynamic Asset Pricing. DAP is a normative theory that grew out of the general equilibrium model of mathematical economics for financial markets, evolved through the capital asset pricing models, and is currently predicated on the absence of arbitrage and on the existence of equivalent martingale measure(s). SPT, by contrast, is applicable under a wide range of assumptions and conditions that may hold in actual equity markets. Unlike dynamic asset pricing, it is consistent with either equilibrium or disequilibrium, with either arbitrage or no-arbitrage, and remains valid regardless of the existence of equivalent martingale measure(s).

This survey reviews the central ideas of SPT and presents examples of portfolios and markets with a wide variety of different properties. SPT is a fast-evolving field, so we also present a number of research problems that remain open, at least at the time of this writing. Proofs for some of the results are included here, but at other times simply a reference is given.

The survey is divided into four parts. Part I, Basics, introduces the concepts of markets and portfolios, in particular, the *market portfolio*, that most important portfolio of them all. In this first part we also encounter the *excess growth rate* process, a quantity that pervades SPT. Part II, Diversity & Arbitrage, introduces market *diversity* and shows how diversity can lead to *relative arbitrage* in an equity market. Historically, these were among the first phenomena analyzed using SPT. *Portfolio generating functions* are versatile tools for constructing portfolios with particular properties, and these functions are discussed in Part III, Functionally Generated Portfolios. Here we also consider stocks identified by rank, as opposed to by name, and discuss implications regarding the *size effect*. Roughly speaking, these first three parts of the survey outline the techniques that historically have comprised SPT; the fourth part looks toward the future.

Part IV, Abstract Markets, is devoted to the area of much of the current research in SPT. *Abstract markets* are models of equity markets that exhibit certain characteristics of real stock markets, but for which the precise mathematical structure is known (since we can define them as we wish!). Here we see *volatility-stabilized* markets that are not diverse but nevertheless allow arbitrage, and we also look at *rank-based* markets that have stability properties similar to those of real stock markets. Several problems regarding these abstract markets are proposed.

Part I
Basics

SPT uses the *logarithmic representation* for stocks and portfolios rather than the *arithmetic representation* used in "classical" mathematical finance. In the logarithmic representation, the classical *rate of return* is replaced by the *growth rate*, sometimes referred to as the *geometric rate of return* or the *logarithmic rate of return*. The logarithmic and arithmetic representations are equivalent, but nevertheless the different perspectives bring to light distinct aspects of portfolio behavior. The use of the logarithmic representation in no way implies the use of a logarithmic utility function: indeed, SPT is not concerned with expected utility maximization at all.

We introduce here the basic structures of SPT, *stocks* and *portfolios*, and discuss that most important portfolio of them all: the *market portfolio*. We show that the growth rate of a portfolio depends not only on the growth rates of the component stocks, but also on the *excess growth rate*, which is determined by the stocks' variances and covariances. Finally, we consider a few optimization problems in the logarithmic setting.

Most of the material in this part can be found in Fernholz (2002).

1 Markets and Portfolios

We shall place ourselves in a model $\mathcal{M}$ for a financial market of the form

$$\begin{aligned} dB(t) &= B(t)r(t)\,dt, \qquad\qquad B(0)=1, \\ dX_i(t) &= X_i(t)\Big(b_i(t)dt + \sum_{\nu=1}^{d} \sigma_{i\nu}(t)\,dW_\nu(t)\Big), \qquad X_i(0)=x_i>0,\ i=1,\ldots,n, \end{aligned} \tag{1.1}$$

consisting of a money-market $B(\cdot)$ and of n stocks, whose prices $X_1(\cdot),\cdots,X_n(\cdot)$ are driven by the $d-$dimensional Brownian motion $W(\cdot)=\big(W_1(\cdot),\cdots,W_d(\cdot)\big)'$ with $d\geq n$. Contrary to a usual assumption imposed on such models, here it is not crucial that the filtration $\mathbb{F}=\{\mathcal{F}(t)\}_{0\leq t\leq T}$ be the one generated by the Brownian motion itself. Thus, and until further notice, we shall take $\mathbb{F}$ to contain (possibly strictly) this Brownian filtration $\mathbb{F}^W=\{\mathcal{F}^W(t)\}_{0\leq t\leq T}$, where $\mathcal{F}^W(t):=\sigma(W(s),\,0\leq s\leq t)$.

We shall assume that the interest-rate process $r(\cdot)$, the vector-valued process $b(\cdot)=\big(b_1(\cdot),\ldots,b_n(\cdot)\big)'$ of *rates of return*, and the $(m\times d)-$matrix-valued process $\sigma(\cdot)=\big(\sigma_{i\nu}(\cdot)\big)_{1\leq i\leq n,\,1\leq\nu\leq d}$ of *volatilities*, are all $\mathbb{F}-$progressively measurable, and satisfy for every $T\in(0,\infty)$ the integrability conditions

$$\int_0^T |r(t)|\,dt + \sum_{i=1}^n \int_0^T \Big(\big|b_i(t)\big| + \sum_{\nu=1}^d \big(\sigma_{i\nu}(t)\big)^2\Big)\,dt < \infty\,, \qquad \text{a.s.} \tag{1.2}$$

This setting admits a rich class of continuous-path Itô processes, with very general distributions: no Markovian or Gaussian assumption is imposed. In fact, it is possible to extend the scope of the theory to general semimartingale settings; see Kardaras (2003) for details.

With the notation

$$a_{ij}(t) := \sum_{\nu=1}^{d} \sigma_{i\nu}(t)\sigma_{j\nu}(t) = \big(\sigma(t)\sigma'(t)\big)_{ij} = \frac{d}{dt}\langle \log X_i, \log X_j \rangle(t) \tag{1.3}$$

for the non-negative definite matrix-valued *variance/covariance process* $a(\cdot) = \big(a_{ij}(\cdot)\big)_{1\le i,j\le n}$ of the stocks in the market, as well as

$$\gamma_i(t) := b_i(t) - \frac{1}{2}a_{ii}(t), \qquad i = 1, \ldots, n, \tag{1.4}$$

we may use Itô's rule to solve (1.1), in the form

$$d\log X_i(t) \;=\; \gamma_i(t)\,dt + \sum_{\nu=1}^{d} \sigma_{i\nu}(t)\,dW_\nu(t), \quad i = 1, \ldots, n, \tag{1.5}$$

or equivalently:

$$X_i(t) = x_i\,\exp\Big\{ \int_0^t \gamma_i(u)\,du + \sum_{\nu=1}^{d} \int_0^t \sigma_{i\nu}(u)\,dW_\nu(u) \Big\}, \quad 0 \le t < \infty.$$

We shall refer to the quantity of (1.4) as the *growth rate* of the i^{th} stock, because of the a.s. relationship

$$\lim_{T\to\infty} \frac{1}{T}\Big(\log X_i(T) - \int_0^T \gamma_i(t)\,dt \Big) = 0, \qquad i = 1, \cdots, n\,. \tag{1.6}$$

This is valid when the individual stock variances $a_{ii}(\cdot)$ do not increase too quickly, e.g., if we have $\lim_{t\to\infty} t^{-1}a_{ii}(t)\log\log t = 0$, a.s.; then (1.6) follows from the law of the iterated logarithm and from the representation of (local) martingales as time-changed Brownian motions.

1.1 Definition. A *portfolio* $\pi(\cdot) = \big(\pi_1(\cdot), \ldots, \pi_n(\cdot)\big)'$ is an $\mathbb{F}-$progressively measurable process, bounded uniformly in (t, ω), with values in the set

$$\big\{ (\pi_1, \ldots, \pi_n) \in \mathbb{R}^n \,\big|\, \pi_1 + \cdots + \pi_n = 1 \big\}.$$

A *long-only portfolio* $\pi(\cdot) = \big(\pi_1(\cdot), \ldots, \pi_n(\cdot)\big)'$ is a portfolio that takes values in the set

$$\Delta^n := \big\{ (\pi_1, \ldots, \pi_n) \in \mathbb{R}^n \,\big|\, \pi_1 \ge 0, \ldots, \pi_n \ge 0 \quad \text{and} \quad \pi_1 + \cdots + \pi_n = 1 \big\}.$$

For future reference, we shall introduce also the notation

$$\Delta^n_+ := \big\{ (\pi_1, \ldots, \pi_n) \in \Delta^n \,|\, \pi_1 > 0, \ldots, \pi_n > 0 \big\}. \qquad \square$$

Thus, a portfolio can sell one or more stocks short (though certainly not all) but is never allowed to borrow from, or invest in, the money market. A long-only portfolio, of course, sells no stocks short at all. The interpretation is that $\pi_i(t)$ represents the proportion of wealth $V(t) \equiv V^{w,\pi}(t)$ invested at time t in the i^{th} stock, so the quantities

$$h_i(t) = \pi_i(t)V^{w,\pi}(t), \qquad i = 1, \cdots, n \tag{1.7}$$

are the amounts invested at any given time t in the individual stocks.

The wealth process $V^{w,\pi}(\cdot)$, that corresponds to a portfolio $\pi(\cdot)$ and initial capital $w > 0$, satisfies the stochastic equation

$$\begin{aligned} \frac{dV^{w,\pi}(t)}{V^{w,\pi}(t)} &= \sum_{i=1}^{n} \pi_i(t)\,\frac{dX_i(t)}{X_i(t)} \;=\; \pi'(t)\big[\,b(t)dt + \sigma(t)\,dW(t)\,\big] \\ &= b_\pi(t)\,dt + \sum_{\nu=1}^{d} \sigma_{\pi\nu}(t)\,dW_\nu(t), \quad V^{w,\pi}(0) = w, \end{aligned} \tag{1.8}$$

and

$$b_\pi(t) := \sum_{i=1}^n \pi_i(t) b_i(t), \qquad \sigma_{\pi\nu}(t) := \sum_{i=1}^n \pi_i(t)\sigma_{i\nu}(t) \quad \text{for} \quad \nu = 1, \ldots, d\,. \tag{1.9}$$

These quantities are, respectively, the rate-of-return and the volatility coëfficients that correspond to the portfolio $\pi(\cdot)$.

By analogy with (1.5) we can write the solution of the equation (1.9) as

$$d\log V^{w,\pi}(t) = \gamma_\pi(t)\,dt + \sum_{\nu=1}^d \sigma_{\pi\nu}(t)\,dW_\nu(t), \qquad V^{w,\pi}(0) = w\,, \tag{1.10}$$

or equivalently

$$V^{w,\pi}(t) = w\,\exp\Big\{\int_0^t \gamma_\pi(u)\,du + \sum_{\nu=1}^d \int_0^t \sigma_{\pi\nu}(u)\,dW_\nu(u)\Big\}, \quad 0 \le t < \infty\,.$$

Here

$$\gamma_\pi(t) := \sum_{i=1}^n \pi_i(t)\gamma_i(t) + \gamma_\pi^*(t) \tag{1.11}$$

is the *growth rate*, and

$$\gamma_\pi^*(t) := \frac{1}{2}\Big(\sum_{i=1}^n \pi_i(t)a_{ii}(t) - \sum_{i=1}^n\sum_{j=1}^n \pi_i(t)a_{ij}(t)\pi_j(t)\Big) \tag{1.12}$$

the *excess growth rate*, of the portfolio $\pi(\cdot)$. As we shall see in Lemma 3.3 below, for a long-only portfolio this excess growth rate is always non-negative – and is strictly positive for such portfolios that do not concentrate their holdings in just one stock.

Again, the terminology "growth rate" is justified by the a.s. property

$$\lim_{T\to\infty} \frac{1}{T}\Big(\log V^{w,\pi}(T) - \int_0^T \gamma_\pi(t)\,dt\Big) = 0, \tag{1.13}$$

valid when all eigenvalues of the variance/covariance matrix process $a(\cdot)$ of (1.3) are bounded: i.e., when

$$\xi' a(t)\xi = \xi'\sigma(t)\sigma'(t)\xi \le K\|\xi\|^2, \quad \forall \;\; t \in [0,\infty) \quad \text{and} \quad \xi \in \mathbb{R}^n \tag{1.14}$$

holds almost surely, for some constant $K \in (0,\infty)$. We shall refer to (1.14) as the *uniform boundedness* condition on the volatility structure of $\mathcal{M}$.

Without further comment we shall write $V^\pi(\cdot) \equiv V^{1,\pi}(\cdot)$ for initial wealth $w = \$1$. Let us also note the following analogue of (1.10), namely

$$d\log V^\pi(t) = \gamma_\pi^*(t)\,dt + \sum_{i=1}^n \pi_i(t)\,d\log X_i(t). \tag{1.15}$$

1.2 Definition. We shall use the reverse-order-statistics notation for the weights of a portfolio $\pi(\cdot)$, ranked at time t in decreasing order, from largest down to smallest:

$$\max_{1\le i\le n} \pi_i(t) =: \pi_{(1)}(t) \ge \pi_{(2)}(t) \ge \ldots \ge \pi_{(n-1)}(t) \ge \pi_{(n)}(t) := \min_{1\le i\le n} \pi_i(t). \tag{1.16}$$

1.1 General Trading Strategies

For completeness of exposition and for later use in this subsection, let us go briefly beyond portfolios and recall the notion of *trading strategies:* these are allowed to invest in (or borrow from) the money market. Formally, they are $\mathbb{F}-$progressively measurable, $\mathbb{R}^n-$valued processes $h(\cdot) = \big(h_1(\cdot), \cdots h_n(\cdot)\big)'$ that satisfy the integrability condition

$$\sum_{i=1}^{n} \int_0^T \Big(\big|h_i(t)\big|\big|b_i(t) - r(t)\big| + h_i^2(t) a_{ii}(t) \Big)\, dt < \infty, \qquad \text{a.s.}$$

for every $T \in (0, \infty)$. The interpretation is that the real-valued random variable $h_i(t)$ stands for the dollar amount invested by $h(\cdot)$ at time t in the i^{th} stock. If we denote by $\mathcal{V}^{w,h}(t)$ the wealth at time t corresponding to this strategy $h(\cdot)$ and to an initial capital $w > 0$, then $\mathcal{V}^{w,h}(t) - \sum_{i=1}^n h_i(t)$ is the amount invested in the money market, and we have the dynamics

$$d\mathcal{V}^{w,h}(t) = \Big(\mathcal{V}^{w,h}(t) - \sum_{i=1}^{n} h_i(t)\Big) r(t)\, dt + \sum_{i=1}^{n} h_i(t) \Big(b_i(t)\, dt + \sum_{\nu=1}^{d} \sigma_{i\nu}(t)\, dW_\nu(t)\Big),$$

or equivalently,

$$\frac{\mathcal{V}^{w,h}(t)}{B(t)} = w + \int_0^t \frac{h'(s)}{B(s)} \Big(\big(b(s) - r(s)\mathbf{I}\big)\, ds + \sigma(s)\, dW(s)\Big), \quad 0 \le t \le T. \tag{1.17}$$

Here $\mathbf{I} = (1, \cdots, 1)'$ is the $n-$dimensional column vector with 1 in all entries. Again without further comment, we shall write $\mathcal{V}^h(\cdot) \equiv \mathcal{V}^{1,h}(\cdot)$ for initial wealth $w = \$1$.

As mentioned already, all quantities $h_i(\cdot),\ 1 \le i \le n$ and $\mathcal{V}^{w,h}(t) - h'(\cdot)\mathbf{I}$ are allowed to take negative values. This possibility opens the door to the notorious *doubling strategies* of martingale theory (e.g. [KS] (1998), Chapter 1). In order to rule these out, we shall confine ourselves here to trading strategies $h(\cdot)$ that satisfy

$$\mathbb{P}\left(\mathcal{V}^{w,h}(t) \ge 0, \quad \forall\ 0 \le t \le T\right) = 1\,. \tag{1.18}$$

Such strategies will be called *admissible* for the initial capital $w > 0$ on the time horizon $[0,T]$; their collection will be denoted $\mathcal{H}(w;T)$. We shall also find useful to look at the collection $\mathcal{H}_+(w;T) \subset \mathcal{H}(w;T)$ of *strongly admissible* strategies, with $\mathbb{P}\big(\mathcal{V}^{w,h}(t) > 0, \quad \forall\ 0 \le t \le T\big) = 1$.

Each portfolio $\pi(\cdot)$ generates, via (1.7), a trading strategy $h(\cdot) \in \mathcal{H}_+(w) := \bigcap_{T>0} \mathcal{H}_+(w;T)$; and we have $\mathcal{V}^{w,h}(\cdot) \equiv V^{w,\pi}(\cdot)$. It is not difficult to see from (1.8) that the trading strategy generated by a portfolio $\pi(\cdot)$ is *self-financing* (see, e.g., Duffie (1992) for a discussion).

2 The Market Portfolio

Suppose we normalize so that each stock has always just one share outstanding; then the stock price $X_i(t)$ can be interpreted as the capitalization of the i^{th} company at time t, and the quantities

$$X(t) := X_1(t) + \ldots + X_n(t) \qquad \text{and} \qquad \mu_i(t) := \frac{X_i(t)}{X(t)}\,, \quad i = 1, \ldots, n \tag{2.1}$$

as the total capitalization of the market and the relative capitalizations of the individual companies, respectively. Clearly $0 < \mu_i(t) < 1,\ \ \forall\ i = 1, \ldots, n$ and $\sum_{i=1}^n \mu_i(t) = 1$, so we may think of the vector process $\mu(\cdot) = \big(\mu_1(\cdot), \ldots, \mu_n(\cdot)\big)'$ as a portfolio that invests the proportion $\mu_i(t)$ of current wealth in the i^{th} asset at all times. The resulting wealth process $V^{w,\mu}(\cdot)$ satisfies

$$\frac{dV^{w,\mu}(t)}{V^{w,\mu}(t)} \;=\; \sum_{i=1}^{n} \mu_i(t)\, \frac{dX_i(t)}{X_i(t)} \;=\; \sum_{i=1}^{n} \frac{dX_i(t)}{X(t)} \;=\; \frac{dX(t)}{X(t)}\,,$$

in accordance with (2.1) and (1.8). In other words,

$$V^{w,\mu}(\cdot) \equiv \frac{w}{X(0)}\, X(t)\,; \tag{2.2}$$

investing in the portfolio $\mu(\cdot)$ is tantamount to ownership of the entire market, in proportion of course to the initial investment. For this reason, we shall call $\mu(\cdot)$ of (2.1) the *market portfolio.* By analogy with (1.10) we have

$$d\log V^{w,\mu}(t) = \gamma_\mu(t)\,dt + \sum_{\nu=1}^{d} \sigma_{\mu\nu}(t)\,dW_\nu(t), \qquad V^{w,\mu}(0) = w, \tag{2.3}$$

and comparison of this last equation (2.3) with (1.5) gives the dynamics of the market-weights

$$d\log\mu_i(t) = \big(\gamma_i(t) - \gamma_\mu(t)\big)\,dt + \sum_{\nu=1}^{d} \big(\sigma_{i\nu}(t) - \sigma_{\mu\nu}(t)\big)\,dW_\nu(t) \tag{2.4}$$

in (2.1) for all stocks $i = 1, \ldots, n$ in the notation of (1.9), (1.11); equivalently,

$$\frac{d\mu_i(t)}{\mu_i(t)} \;=\; \Big(\gamma_i(t) - \gamma_\mu(t) + \frac{1}{2}\tau^\mu_{ii}(t)\Big)\,dt + \sum_{\nu=1}^{d} \big(\sigma_{i\nu}(t) - \sigma_{\mu\nu}(t)\big)\,dW_\nu(t)\,. \tag{2.5}$$

Here we introduce, for an arbitrary portfolio $\pi(\cdot)$ and with e_i denoting the i^{th} unit vector in $\mathbb{R}^n$, the quantities

$$\begin{aligned} \tau^\pi_{ij}(t) \;&:= \sum_{\nu=1}^{d} \big(\sigma_{i\nu}(t) - \sigma_{\pi\nu}(t)\big)\big(\sigma_{j\nu}(t) - \sigma_{\pi\nu}(t)\big) \\ &= \big(\pi(t) - e_i\big)' a(t)\big(\pi(t) - e_j\big) = a_{ij}(t) - a_{\pi i}(t) - a_{\pi j}(t) + a_{\pi\pi}(t) \end{aligned} \tag{2.6}$$

for $1 \le i, j \le n$, and set

$$a_{\pi i}(t) := \sum_{j=1}^{n} \pi_j(t) a_{ij}(t), \qquad a_{\pi\pi}(t) := \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i(t) a_{ij}(t)\pi_j(t) = \sum_{\nu=1}^{d} \big(\sigma_{\pi\nu}(t)\big)^2\,. \tag{2.7}$$

It is seen from (1.10) that this quantity is the variance of the portfolio $\pi(\cdot)$.

We shall call the matrix-valued process $\tau^\pi(\cdot) = \big(\tau^\pi_{ij}(\cdot)\big)_{1\le i,j\le n}$ of (2.6) the *variance/covariance process of individual stocks relative to the portfolio* $\pi(\cdot)$. It satisfies the elementary property

$$\sum_{j=1}^{n} \tau^\pi_{ij}(t)\pi_j(t) = 0, \quad i = 1, \cdots, n. \tag{2.8}$$

The corresponding quantities

$$\tau^\mu_{ij}(t) := \sum_{\nu=1}^{d} \big(\sigma_{i\nu}(t) - \sigma_{\mu\nu}(t)\big)\big(\sigma_{j\nu}(t) - \sigma_{\mu\nu}(t)\big) = \frac{d\langle \mu_i, \mu_j\rangle(t)}{\mu_i(t)\mu_j(t)dt}, \quad 1 \le i, j \le n \tag{2.9}$$

of (2.6) for the market portfolio $\pi(\cdot) \equiv \mu(\cdot)$, are the variances/covariances of the individual stocks relative to the entire market. (For the second equality in (2.9), we have used the semimartingale decomposition of (2.5).)

3 Some Useful Properties

In this section we collect together some useful properties of the relative variance/covariance process in (2.6), for ease of reference in future usage. For any given stock i and portfolio $\pi(\cdot)$, the *relative return process* of the i^{th} stock versus $\pi(\cdot)$ is the process

$$R_i^\pi(t) := \log\left(\frac{X_i(t)}{V^{w,\pi}(t)}\right)\Bigg|_{w=X_i(0)}, \qquad 0 \le t < \infty. \tag{3.1}$$

3.1 Lemma. *For any portfolio $\pi(\cdot)$, and for all $1 \le i,j \le n$ and $t \in [0,\infty)$, we have, almost surely:*

$$\tau_{ij}^\pi(t) = \frac{d}{dt}\langle R_i^\pi, R_j^\pi\rangle(t), \qquad \textit{in particular,} \qquad \tau_{ii}^\pi(t) = \frac{d}{dt}\langle R_i^\pi\rangle(t) \ge 0, \tag{3.2}$$

and the matrix $\tau^\pi(t) = \big(\tau_{ij}^\pi(t)\big)_{1\le i,j\le n}$ is a.s. nonnegative definite. Furthermore, if the variance/covariance matrix $a(t)$ is positive definite, then the matrix $\tau^\pi(t)$ has rank $n-1$, and its null space is spanned by the vector $\pi(t)$, almost surely.

Proof: Comparing (1.5) with (1.10) we get the analogue

$$dR_i^\pi(t) = \big(\gamma_i(t) - \gamma_\pi(t)\big)\,dt + \sum_{\nu=1}^{d}\big(\sigma_{i\nu}(t) - \sigma_{\pi\nu}(t)\big)\,dW_\nu(t),$$

of (2.4), from which the first two claims follow.

Now suppose that $a(t)$ is positive definite. For any $x \in \mathbb{R}^n \setminus \{0\}$ and with $\eta := \sum_{i=1}^n x_i$, we compute from (2.4):

$$x'\tau^\pi(t)x = x'a(t)x - 2\eta x'a(t)\pi(t) + \eta^2\pi'(t)a(t)\pi(t).$$

If $\sum_{i=1}^n x_i = 0$, then $x'\tau^\pi(t)x = x'a(t)x > 0$. If on the other hand $\eta := \sum_{i=1}^n x_i \neq 0$, we consider the vector $y := x/\eta$ that satisfies $\sum_{i=1}^n y_i = 1$, and observe that $\eta^{-2}x'\tau^\pi(t)x$ is equal to

$$y'\tau^\pi(t)y = y'a(t)y - 2y'a(t)\pi(t) + \pi'(t)a(t)\pi(t) = \big(y - \pi(t)\big)'a(t)\big(y-\pi(t)\big),$$

thus zero if and only if $y = \pi(t)$, or equivalently $x = \eta\pi(t)$. □

3.2 Lemma. *For any two portfolios $\pi(\cdot)$, $\rho(\cdot)$, we have*

$$d\,\log\left(\frac{V^\pi(t)}{V^\rho(t)}\right) = \gamma_\pi^*(t)\,dt + \sum_{i=1}^n \pi_i(t)\,d\log\left(\frac{X_i(t)}{V^\rho(t)}\right). \tag{3.3}$$

In particular, we get the dynamics

$$\begin{aligned} d\,\log\left(\frac{V^\pi(t)}{V^\mu(t)}\right) &= \gamma_\pi^*(t)\,dt + \sum_{i=1}^n \pi_i(t)\,d\log\mu_i(t) \\ &= \big(\gamma_\pi^*(t) - \gamma_\mu^*(t)\big)\,dt + \sum_{i=1}^n\big(\pi_i(t) - \mu_i(t)\big)\,d\log\mu_i(t) \end{aligned} \tag{3.4}$$

for the relative return of an arbitrary portfolio $\pi(\cdot)$ with respect to the market.

Proof: The equation (3.3) follows from (1.15), and the first equality in (3.4) is the special case of (3.3) with $\rho(\cdot) \equiv \mu(\cdot)$. The second equality in (3.4) follows upon observing from (2.4), that

$$\sum_{i=1}^n \mu_i(t)\,d\log\mu_i(t) = \sum_{i=1}^n \mu_i(t)(\gamma_i(t) - \gamma_\mu(t))\,dt = -\gamma_\mu^*(t)\,dt.$$

3.3 Lemma. *For any two portfolios* $\pi(\cdot)$, $\rho(\cdot)$ *we have the* numéraire-invariance *property*

$$\gamma_\pi^*(t) = \frac{1}{2}\Big(\sum_{i=1}^n \pi_i(t)\tau_{ii}^\rho(t) - \sum_{i=1}^n\sum_{j=1}^n \pi_i(t)\pi_j(t)\tau_{ij}^\rho(t)\Big). \tag{3.5}$$

In particular, recalling (2.8), we obtain the representation

$$\gamma_\pi^*(t) = \frac{1}{2}\sum_{i=1}^n \pi_i(t)\tau_{ii}^\pi(t) \tag{3.6}$$

for the excess growth rate, as a weighted average of the individual stocks' variances $\tau_{ii}^\pi(\cdot)$ *relative to* $\pi(\cdot)$*, as in (2.6). Whereas, from (3.6), (3.2) and Definition 1.1, we get for any long-only portfolio* $\pi(\cdot)$ *the property:*

$$\gamma_\pi^*(t) \geq 0\ . \tag{3.7}$$

Proof: From (2.6) we get

$$\sum_{i=1}^n \pi_i(t)\tau_{ii}^\rho(t) = \sum_{i=1}^n \pi_i(t)a_{ii}(t) - 2\sum_{i=1}^n \pi_i(t)a_{\rho i}(t) + a_{\rho\rho}(t)$$

as well as

$$\sum_{i=1}^n\sum_{j=1}^n \pi_i(t)\tau_{ij}^\rho(t)\pi_j(t) \;=\; \sum_{i=1}^n\sum_{j=1}^n \pi_i(t)a_{ij}(t)\pi_j(t) - 2\sum_{i=1}^n \pi_i(t)a_{\rho i}(t) + a_{\rho\rho}(t),$$

and (3.5) follows from (1.12). $\square$

For the market portfolio, equation (3.6) becomes

$$\gamma_\mu^*(t) = \frac{1}{2}\sum_{i=1}^n \mu_i(t)\tau_{ii}^\mu(t)\,; \tag{3.8}$$

the summation on the right-hand-side is the average, according to the market weights of individual stocks, of these stocks' variances relative to the market. Thus, (3.8) gives an interpretation of the excess growth rate of the market portfolio, as a measure of the market's "intrinsic" volatility.

3.4 Remark. Note that (3.4), in conjunction with (2.4), (2.5) and the numéraire-invariance property (3.5), implies that for any portfolio $\pi(\cdot)$ we have the *relative return* formula

$$d\,\log\left(\frac{V^\pi(t)}{V^\mu(t)}\right) \;=\; \sum_{i=1}^n \frac{\pi_i(t)}{\mu_i(t)}\,d\mu_i(t) - \frac{1}{2}\left(\sum_{i=1}^n\sum_{j=1}^n \pi_i(t)\pi_j(t)\tau_{ij}^\mu(t)\right)dt\,. \tag{3.9}$$

3.5 Lemma. *Assume that the variance/covariance process* $a(\cdot)$ *of (1.3) satisfies the following* strong non-degeneracy condition: *all its eigenvalues are bounded away from zero. To wit, assume there exists a constant* $\varepsilon \in (0,\infty)$ *such that*

$$\xi' a(t)\xi = \xi'\sigma(t)\sigma'(t)\xi \geq \varepsilon\|\xi\|^2, \quad \forall\;\; t\in[0,\infty) \quad \text{and} \quad \xi\in\mathbb{R}^n \tag{3.10}$$

holds almost surely. Then for every portfolio $\pi(\cdot)$ *and* $0 \leq t < \infty$*, we have in the notation of (1.16) the inequalities*

$$\varepsilon\big(1-\pi_i(t)\big)^2 \leq \tau_{ii}^\pi(t), \qquad i = 1,\cdots,n\,, \tag{3.11}$$

almost surely. If the portfolio $\pi(\cdot)$ *is long-only, we have also*

$$\frac{\varepsilon}{2}\big(1-\pi_{(1)}(t)\big) \leq \gamma_\pi^*(t). \tag{3.12}$$

Proof: With e_i denoting the i^{th} unit vector in $\mathbb{R}^n$, we have

$$\tau_{ii}^{\pi}(t) = (\pi(t) - e_i)' a(t)(\pi(t) - e_i) \geq \varepsilon \|\pi(t) - e_i\|^2 = \varepsilon \Big(\big(1 - \pi_i(t)\big)^2 + \sum_{j \neq i} \pi_j^2(t) \Big)$$

from (2.6) and (3.10), and (3.11) follows. Back into (3.6), and with $\pi_i(t) \geq 0 \; \forall \; i = 1, \cdots, n$, this lower estimate gives

$$\begin{aligned}
\gamma_*^{\pi}(t) &\geq \frac{\varepsilon}{2} \sum_{i=1}^{n} \pi_i(t) \Big(\big(1 - \pi_i(t)\big)^2 + \sum_{j \neq i} \pi_j^2(t) \Big) \\
&= \frac{\varepsilon}{2} \Big(\sum_{i=1}^{n} \pi_i(t) \big(1 - \pi_i(t)\big)^2 + \sum_{j=1}^{n} \pi_j^2(t) \big(1 - \pi_j(t)\big) \Big) \\
&= \frac{\varepsilon}{2} \sum_{i=1}^{n} \pi_i(t) \big(1 - \pi_i(t)\big) \geq \frac{\varepsilon}{2} \big(1 - \pi_{(1)}(t)\big). \qquad \square
\end{aligned}$$

3.6 Lemma. *Assume that the uniform boundedness condition (1.14) holds; then for every long-only portfolio $\pi(\cdot)$, and for $0 \leq t < \infty$, we have in the notation of (1.16) the a.s. inequalities*

$$\tau_{ii}^{\pi}(t) \leq K \big(1 - \pi_i(t)\big) \big(2 - \pi_i(t)\big), \qquad i = 1, \cdots, n \tag{3.13}$$

$$\gamma_{\pi}^{*}(t) \leq 2K \big(1 - \pi_{(1)}(t)\big). \tag{3.14}$$

Proof: As in the previous proof, we get

$$\begin{aligned}
\tau_{ii}^{\pi}(t) &\leq K \Big(\big(1 - \pi_i(t)\big)^2 + \sum_{j \neq i} \pi_j^2(t) \Big) \\
&\leq K \Big(\big(1 - \pi_i(t)\big)^2 + \sum_{j \neq i} \pi_j(t) \Big) = K(1 - \pi_i(t))(2 - \pi_i(t))
\end{aligned}$$

as claimed in (3.13), and bringing this estimate into (3.6) leads to

$$\begin{aligned}
\gamma_*^{\pi}(t) &\leq K \sum_{i=1}^{n} \pi_i(t) \big(1 - \pi_i(t)\big) \\
&= K \Big(\pi_{(1)}(t) \big(1 - \pi_{(1)}(t)\big) + \sum_{k=2}^{n} \pi_{(k)}(t) \big(1 - \pi_{(k)}(t)\big) \Big) \\
&\leq K \Big(\big(1 - \pi_{(1)}(t)\big) + \sum_{k=2}^{n} \pi_{(k)}(t) \Big) = 2K \big(1 - \pi_{(1)}(t)\big). \qquad \square
\end{aligned}$$

4 Portfolio Optimization

We can already formulate some interesting optimization problems.

Problem 1: Quadratic criterion, linear constraint (Markowitz, 1952). Minimize the portfolio variance $a_{\pi\pi}(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i(t) a_{ij}(t) \pi_j(t)$, among all portfolios $\pi(\cdot)$ with rate-of-return $b_{\pi}(t) = \sum_{i=1}^{n} \pi_i(t) b_i(t) \geq b_0$ at least equal to a given constant.

Problem 2: Quadratic criterion, quadratic constraint. Minimize the portfolio variance

$$a_{\pi\pi}(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i(t) a_{ij}(t) \pi_j(t)$$

among all portfolios $\pi(\cdot)$ with growth-rate at least equal to a given constant γ_0:

$$\sum_{i=1}^n \pi_i(t) \left(\gamma_i(t) + \frac{1}{2} a_{ii}(t) \right) \geq \gamma_0 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \pi_i(t) a_{ij}(t) \pi_j(t).$$

Problem 3: *Maximize the probability of reaching a given "ceiling" $\mathfrak{c}$ before reaching a given "floor" $\mathfrak{f}$, with $0 < \mathfrak{f} < w < \mathfrak{c} < \infty$.* More specifically, maximize $\mathbb{P}[\,\mathfrak{T}_{\mathfrak{c}} < \mathfrak{T}_{\mathfrak{f}}\,]$, with $\mathfrak{T}_\xi := \inf\{t \geq 0 \,|\, V^{w,\pi}(t) = \xi\}$ for $\xi \in (0, \infty)$.

In the case of constant coëfficients γ_i and a_{ij}, the solution to this problem comes in the following simple form: one looks at the mean-variance, or *signal-to-noise*, ratio

$$\frac{\gamma_\pi}{a_{\pi\pi}} = \frac{\sum_{i=1}^n \pi_i(\gamma_i + \frac{1}{2} a_{ii})}{\sum_{i=1}^n \sum_{j=1}^n \pi_i a_{ij} \pi_j} - \frac{1}{2},$$

and finds a portfolio π that maximizes it (Pestien & Sudderth, 1985).

Problem 4: *Minimize the expected time $\mathbb{E}(\mathfrak{T}_{\mathfrak{c}})$ until a given "ceiling" $\mathfrak{c} \in (w, \infty)$ is reached.*

Again with constant coëfficients, it turns out that it is enough to try and maximize the drift in the equation for $\log V^{w,\pi}(\cdot)$, namely

$$\gamma_\pi = \sum_{i=1}^n \pi_i \left(\gamma_i + \tfrac{1}{2} a_{ii} \right) - \tfrac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \pi_i a_{ij} \pi_j,$$

the portfolio growth-rate (Heath, Orey, Pestien & Sudderth, 1987).

Problem 5: *Maximize the probability $\mathbb{P}[\mathfrak{T}_{\mathfrak{c}} < T \wedge \mathfrak{T}_{\mathfrak{f}}]$ of reaching a given "ceiling" $\mathfrak{c}$ before reaching a given "floor" $\mathfrak{f}$ with $0 < \mathfrak{f} < w < \mathfrak{c} < \infty$, by a given "deadline" $T \in (0, \infty)$.*

Always with constant coëfficients, suppose there is a portfolio $\hat{\pi} = (\hat{\pi}_1, \ldots, \hat{\pi}_n)'$ that maximizes *both* the signal-to-noise ratio *and* the variance,

$$\frac{\gamma_\pi}{a_{\pi\pi}} = \frac{\sum_{i=1}^n \pi_i(\gamma_i + \frac{1}{2} a_{ii})}{\sum_{i=1}^n \sum_{j=1}^n \pi_i a_{ij} \pi_j} - \frac{1}{2} \qquad \text{and} \qquad a_{\pi\pi} = \sum_{i=1}^n \sum_{j=1}^n \pi_i a_{ij} \pi_j,$$

respectively, over all $\pi_1 \geq 0, \ldots, \pi_n \geq 0$ with $\sum_{i=1}^n \pi_i = 1$. Then this portfolio $\hat{\pi}$ is optimal for the above criterion (Sudderth & Weerasinghe, 1989).

This is a big assumption; it is satisfied, for instance, under the (very stringent, and unnatural...) condition that, for some $G > 0$, we have

$$\gamma_i + \frac{1}{2} a_{ii} = -G, \quad \text{for all} \;\; i = 1, \ldots, n.$$

As far as the authors are aware, nobody seems to have solved this problem when such simultaneous maximization is not possible.

Part II
Diversity & Arbitrage

Roughly speaking, a market is *diverse* if it avoids concentration of all its capital into a single stock, and the *diversity* of a market is a measure of how uniformly the capital is spread among the stocks. These concepts were introduced in Fernholz (1999), and it was shown in Fernholz (2002), Section 3.3, and in Fernholz, Karatzas & Kardaras (2005) that market diversity gives rise to arbitrage.

Unlike classical mathematical finance, SPT is not averse to the existence of arbitrage in markets, but rather studies market characteristics that imply the existence of arbitrage. Moreover, the existence of arbitrage does not preclude the development of option pricing theory or certain types of utility maximization. These and other related ideas are presented in this part of the survey.

5 Diversity

The notion of diversity for a financial market corresponds to the intuitive (and descriptive) idea, that no single company can ever be allowed to dominate the entire market in terms of relative capitalization. To make this notion precise, let us say that the model $\mathcal{M}$ of (1.1), (1.2) is *diverse* on the time horizon $[0, T]$, if there exists a number $\delta \in (0, 1)$ such that the quantities of (2.1) satisfy almost surely

$$\mu_{(1)}(t) < 1 - \delta, \quad \forall \ 0 \le t \le T \tag{5.1}$$

in the order-statistics notation of (1.16). In a similar vein, we say that $\mathcal{M}$ is *weakly diverse* on the time horizon $[0, T]$, if for some $\delta \in (0, 1)$ we have

$$\frac{1}{T}\int_0^T \mu_{(1)}(t)dt < 1 - \delta \tag{5.2}$$

almost surely. We say that $\mathcal{M}$ is *uniformly weakly diverse* on $[T_0, \infty)$, if there exists a $\delta \in (0, 1)$ such that (5.2) holds a.s. for every $T \in [T_0, \infty)$.

It follows directly from (3.14) of Lemma 3.6 that, under the condition (1.14), the model $\mathcal{M}$ of (1.1), (1.2) is diverse (respectively, weakly diverse) on the time interval $[0, T]$, if there exists a number $\zeta > 0$ such that

$$\gamma^*_\mu(t) \ge \zeta, \quad \forall \ 0 \le t \le T \qquad \Big(\text{respectively,} \quad \frac{1}{T}\int_0^T \gamma^*_\mu(t)\,dt \ge \zeta\Big) \tag{5.3}$$

holds almost surely. And (3.12) of Lemma 3.5 shows that, under the condition (3.10), the conditions of (5.3) are satisfied if diversity (respectively, weak diversity) holds on the time interval $[0, T]$.

As we shall see in section 9, diversity can be ensured by a strongly negative rate of growth for the largest stock, resulting in a sufficiently strong repelling drift (e.g., a log-pole-type singularity) away from an appropriate boundary, as well as non-negative growth rates for all the other stocks.

If all the stocks in $\mathcal{M}$ have the same growth rate ($\gamma_i(\cdot) \equiv \gamma(\cdot)$, $\forall\ 1 \le i \le n$) and (1.14) holds, then we have almost surely:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \gamma^*_\mu(t)\,dt = 0. \tag{5.4}$$

In particular, such an equal-growth-rate market $\mathcal{M}$ cannot be diverse, even weakly, for long time horizons, provided that (3.10) is also satisfied.

Here is a quick argument for these claims: recall that for $X(\cdot) = X_1(\cdot) + \cdots + X_n(\cdot)$ we have

$$\lim_{T\to\infty} \frac{1}{T}\Big(\log X(T) - \int_0^T \gamma_\mu(t)\,dt\Big) = 0, \quad \lim_{T\to\infty} \frac{1}{T}\Big(\log X_i(T) - \int_0^T \gamma(t)\,dt\Big) = 0$$

a.s., from (1.13), (1.6) and $\gamma_i(\cdot) \equiv \gamma(\cdot)$ for all $1 \le i \le n$. But then we have also

$$\lim_{T\to\infty} \frac{1}{T}\Big(\log X_{(1)}(T) - \int_0^T \gamma(t)\, dt\Big) = 0, \quad \text{a.s.}$$

for the biggest stock $X_{(1)}(\cdot) := \max_{1\le i\le n} X_i(\cdot)$, and note the inequalities $X_{(1)}(\cdot) \le X(\cdot) \le nX_{(1)}(\cdot)$. Therefore,

$$\lim_{T\to\infty} \frac{1}{T}\big(\log X_{(1)}(T) - \log X(T)\big) = 0, \quad \text{thus} \quad \lim_{T\to\infty} \frac{1}{T}\int_0^T \big(\gamma_\mu(t) - \gamma(t)\big)\, dt = 0\,,$$

almost surely. But $\gamma_\mu(t) = \sum_{i=1}^n \mu_i(t)\gamma(t) + \gamma^*_\mu(t) = \gamma(t) + \gamma^*_\mu(t)$ because of the assumption of equal growth rates, and (5.4) follows. If (3.10) also holds, then (3.12) and (5.4) imply

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \big(1 - \mu_{(1)}(t)\big)\, dt \;=\; 0$$

almost surely, so weak diversity fails on long time horizons: once in a while a single stock dominates the *entire market*, then recedes; sooner or later another stock takes its place as absolutely dominant leader; and so on.

5.1 Remark. Coherence: We say that the market model $\mathcal{M}$ of (1.1), (1.2) is *coherent*, if the relative capitalizations of (2.1) satisfy

$$\lim_{T\to\infty} \frac{1}{T}\log \mu_i(T) = 0 \qquad \text{almost surely, for each } \; i = 1, \cdots, n \tag{5.5}$$

(i.e., if none of the stocks declines too rapidly). Under the condition (1.14), it can be shown that coherence is equivalent to each of the following two conditions:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \big(\gamma_i(t) - \gamma_\mu(t)\big)\, dt = 0 \qquad \text{a.s., for each } \; i = 1, \cdots, n; \tag{5.6}$$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \big(\gamma_i(t) - \gamma_j(t)\big)\, dt = 0 \quad \text{a.s., for each pair } \; 1 \le i, j \le n. \tag{5.7}$$

5.2 Remark. If all the stocks in the market $\mathcal{M}$ have *constant* (though not necessarily the same) growth rates, and if (1.14), (3.10) hold, then $\mathcal{M}$ cannot be diverse, even weakly, over long time horizons.

6 Relative Arbitrage and Its Consequences

The notion of arbitrage is of paramount importance in mathematical finance. We present in this section an allied notion, that of *relative arbitrage*, and explore some of its consequences. In later sections we shall encounter specific, descriptive conditions on market structure, that lead to this form of arbitrage.

6.1 Definition. Given any two portfolios $\pi(\cdot)$, $\rho(\cdot)$ with the same initial capital $V^\pi(0) = V^\rho(0) = 1$, we shall say that $\pi(\cdot)$ represents an *arbitrage opportunity relative to* $\rho(\cdot)$ over the fixed, finite time horizon $[0, T]$, if

$$\mathbb{P}(V^\pi(T) \ge V^\rho(T)) = 1 \quad \text{and} \quad \mathbb{P}(V^\pi(T) > V^\rho(T)) > 0\,. \tag{6.1}$$

We shall say that $\pi(\cdot)$ represents a *superior long-term growth opportunity*, if

$$\mathcal{L}^{\pi,\rho} := \liminf_{T\to\infty} \frac{1}{T}\log\left(\frac{V^\pi(T)}{V^\rho(T)}\right) > 0 \quad \text{holds a.s.} \tag{6.2}$$

6.2 Remark. The definition of relative arbitrage has historically included the condition that there exist a constant $q = q_{\pi,\rho,T} > 0$ such that

$$\mathbb{P}\big(V^\pi(t) \geq qV^\rho(t),\ \forall\ 0 \leq t \leq T\big) = 1. \tag{6.3}$$

However, if one can find a portfolio $\pi(\cdot)$ that satisfies the domination properties (6.1) relative to some other portfolio $\rho(\cdot)$, then there exists another portfolio $\widehat{\pi}(\cdot)$ that satisfies both (6.3) and (6.1) relative to the same $\rho(\cdot)$. The construction involves a strategy of investing a portion $w \in (0,1)$ of the initial capital \$1 in π, and the remaining proportion $1-w$ in $\rho(\cdot)$. This observation is due to C. Kardaras (2006).

6.1 Strict Local Martingales

Let us place ourselves now, and for the remainder of this section, within the market model $\mathcal{M}$ of (1.1) and under the conditions (1.2), (1.14).

We shall assume further that there exists a *market price of risk* (or "relative risk") $\theta : [0,\infty) \times \Omega \to \mathbb{R}^d$; namely, an $\mathbb{F}-$progressively measurable process with

$$\sigma(t)\theta(t) = b(t) - r(t)\mathbf{I},\ \ \forall\ \ 0 \leq t \leq T \qquad \text{and} \qquad \int_0^T \|\theta(t)\|^2\, dt < \infty \tag{6.4}$$

valid almost surely, for each $T \in (0,\infty)$. (If the volatility matrix $\sigma(\cdot)$ has full rank, namely n, we can take, for instance, $\theta(t) = \sigma'(t)\big(\sigma(t)\sigma'(t)\big)^{-1}[\,b(t) - r(t)\mathbf{I}\,]$ in (6.4).)

In terms of this process $\theta(\cdot)$, we can define the exponential local martingale and supermartingale

$$Z(t) := \exp\Big\{ -\int_0^t \theta'(s)\, dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2\, ds \Big\}, \quad 0 \leq t \leq T \tag{6.5}$$

(a martingale, if and only if $\mathbb{E}(Z(T)) = 1$), and the shifted Brownian Motion

$$\widehat{W}(t) := W(t) + \int_0^t \theta(s)\, ds, \qquad 0 \leq t \leq T. \tag{6.6}$$

6.3 Proposition. A Strict Local Martingale: *Under the assumptions of this subsection, suppose there exists a time horizon $T \in (0,\infty)$ such that relative arbitrage exists on $[0,T]$. Then the process $Z(\cdot)$ of (6.5) is a* strict *local martingale: we have $\mathbb{E}(Z(T)) < 1$.*

Proof: Assume, by way of contradiction, that $\mathbb{E}(Z(T)) = 1$. Then from the Girsanov Theorem (e.g. [KS], section 3.5) the recipe $\mathbb{Q}_T(A) := \mathbb{E}[Z(T)\, 1_A]$ defines an probability measure on $\mathcal{F}(T)$, equivalent to $\mathbb{P}$, under which the process $\widehat{W}(\cdot)$ of (6.6) is Brownian motion.

Under $\mathbb{Q}_T$, the discounted stock prices $X_i(\cdot)/B(\cdot)$, $i = 1, \cdots, n$ are positive martingales, because of

$$d\big(X_i(t)/B(t)\big) \;=\; \big(X_i(t)/B(t)\big) \sum_{\nu=1}^d \sigma_{i\nu}(t)\, d\widehat{W}_\nu(t)$$

and of the uniform boundedness (1.14), which is assumed to hold. As usual, we express this by saying that $\mathbb{Q}_T$ is then an *equivalent martingale measure* (EMM) for the model.

Similarly, for any portfolio $\pi(\cdot)$, we get then from (6.6) and (1.8):

$$d\big(V^\pi(t)/B(t)\big) \;=\; \big(V^\pi(t)/B(t)\big)\, \pi'(t)\sigma(t)\, d\widehat{W}(t), \qquad V^\pi(0) = 1\,; \tag{6.7}$$

and because of condition (1.14), the discounted wealth process $V^\pi(\cdot)/B(\cdot)$ is a positive martingale under $\mathbb{Q}_T$. Thus, the difference $\Delta(\cdot) := (V^\pi(\cdot) - V^\rho(\cdot))/B(\cdot)$ is a martingale under $\mathbb{Q}_T$, for any other portfolio $\rho(\cdot)$ with $V^\rho(0) = 1$. But this gives $\mathbb{E}^{\mathbb{Q}_T}\big(\Delta(T)\big) = \Delta(0) = 0$, a conclusion inconsistent with (6.1) which mandates $\mathbb{Q}_T\big(\Delta(T) \geq 0\big) = 1$ and $\mathbb{Q}_T\big(\Delta(T) > 0\big) > 0$. $\square$

Now let us consider the *deflated stock-price and wealth processes*

$$\widehat{X}_i(t) := \frac{Z(t)}{B(t)}\, X_i(t), \quad i = 1, \cdots, n \qquad \text{and} \qquad \widehat{\mathcal{V}}^{w,h}(t) := \frac{Z(t)}{B(t)}\, \mathcal{V}^{w,h}(t) \tag{6.8}$$

for $0 \le t \le T$, for arbitrary admissible trading strategy $h(\cdot)$ and initial capital $w > 0$. These processes satisfy, respectively, the dynamics

$$d\widehat{X}_i(t) \,=\, \widehat{X}_i(t) \sum_{\nu=1}^{d} \big(\sigma_{i\nu}(t) - \theta_\nu(t)\big)\, dW_\nu(t), \quad \widehat{X}_i(0) = x_i\,, \tag{6.9}$$

$$d\widehat{\mathcal{V}}^{w,h}(t) \,=\, \left(\frac{Z(t)h'(t)}{B(t)} \sigma(t) - \widehat{\mathcal{V}}^{w,h}(t)\theta'(t) \right) dW(t), \quad \widehat{\mathcal{V}}^{w,h}(0) = w \tag{6.10}$$

in conjunction with (1.1), (1.17) and (6.5). In particular, these processes are non-negative local martingales (and supermartingales) under $\mathbb{P}$. (In other words, the ratio $Z(\cdot)/B(\cdot)$ continues to play its usual rôle as deflator of prices in such a market, even when $Z(\cdot)$ is just a local martingale.)

6.4 Remark. Strict Local Martingales Galore: From (6.9), (6.10) it can be shown that, in the setting of Proposition 6.3, the deflated stock-price processes $\widehat{X}_i(\cdot)$ of (6.8) are all *strict* local martingales and (strict) supermartingales:

$$\mathbb{E}\big(\widehat{X}_i(T)\big) < x_i \qquad \text{holds for every } \; i = 1, \cdots, n\,. \tag{6.11}$$

Actually, we shall prove in subsection 6.3 a considerably stronger result: *In the setting of Proposition 6.3, for any given portfolio $\rho(\cdot)$ the process $\widehat{V}^{w,\rho}(\cdot) = Z(\cdot)V^{w,\rho}(\cdot)/B(\cdot)$ of (6.8) is a strict local martingale and (a strict) supermartingale, namely*

$$\mathbb{E}\big(\widehat{V}^{w,\rho}(T)\big) \,<\, w\,. \tag{6.12}$$

6.5 Proposition. Non-Existence of Equivalent Martingale Measure: *In the context of Proposition 6.3, no Equivalent Martingale Measure can exist for the model $\mathcal{M}$ of (1.1), if the filtration is generated by the driving Brownian Motion $W(\cdot)$: $\mathbb{F} = \mathbb{F}^W$.*

Proof: If $\mathbb{F} = \mathbb{F}^W$, and there exists some probability measure $\mathbb{Q}$ which is equivalent to $\mathbb{P}$ on $\mathcal{F}(T)$, then the martingale representation property of the Brownian filtration gives $(d\mathbb{Q}/d\mathbb{P})|_{\mathcal{F}(t)} = Z(t)$, $0 \le t \le T$ for some process $Z(\cdot)$ of the form (6.5) and some progressively measurable $\theta(\cdot)$ with $\int_0^T ||\theta(t)||^2\, dt < \infty$ a.s. Then Itô's rule leads to the extension

$$\frac{d\widehat{X}_i(t)}{\widehat{X}_i(t)} \;=\; \left(b_i(t) - r(t) - \sum_{\nu=1}^{d} \sigma_{i\nu}(t)\theta_\nu(t) \right) dt \;+\; \sum_{\nu=1}^{d} \big(\sigma_{i\nu}(t) - \theta_\nu(t)\big)\, dW_\nu(t)$$

of (6.9) for the deflated stock-prices of (6.8).

But if all the $X_i(\cdot)/B(\cdot)$'s are $\mathbb{Q}-$martingales (that is, if $\mathbb{Q}$ is an equivalent martingale measure), then the $\widehat{X}_i(\cdot)$'s are all $\mathbb{P}-$martingales, and this leads to the first property $\sigma(\cdot)\,\theta(\cdot) = b(\cdot) - r(\cdot)\mathbf{I}$ in (6.4). We repeat now the argument of Proposition 6.3 and arrive at a contradiction with (6.1), the existence of relative arbitrage on $[0, T]$. ☐

6.2 On "Beating the Market"

Let us introduce now the decreasing function

$$f(t) := \frac{1}{X(0)} \cdot \mathbb{E}\left(\frac{Z(t)}{B(t)} X(t) \right), \qquad 0 \le t \le T\,. \tag{6.13}$$

If relative arbitrage exists on the time-horizon $[0, T]$, then we know $f(0) = 1 > f(T) > 0$ from Remark 6.4.

6.6 Remark. With Brownian filtration $\mathbb{F} = \mathbb{F}^W$, $n = d$ and an invertible volatility matrix $\sigma(\cdot)$, consider the maximal relative return

$$\mathfrak{R}(T) := \sup \big\{ r > 0 \,|\, \exists\, h(\cdot) \in \mathcal{H}(1;T) \;\text{ s.t. } \big(\mathcal{V}^h(T)/V^\mu(T) \big) \geq r, \text{ a.s.} \big\} \tag{6.14}$$

in excess of the market, that can be obtained by trading strategies over the interval $[0,T]$. It can be shown easily that this quantity is computed in terms of the function of (6.13), as $\mathfrak{R}(T) = 1/f(T)$.

6.7 Remark. The shortest time to beat the market by a given amount: Let us place ourselves again under the assumptions of Remark 6.6, but now assume that relative arbitrage exists on $[0,T]$ for *every* $T \in (0,\infty)$; see section 8 for elaboration. For a given "exceedance level" $r > 1$, consider the shortest length of time

$$\mathbf{T}(r) := \inf \big\{ T > 0 \,|\, \exists\, h(\cdot) \in \mathcal{H}(1;T) \;\text{ s.t. } \big(\mathcal{V}^h(T)/V^\mu(T) \big) \geq r, \text{ a.s.} \big\} \tag{6.15}$$

required to guarantee a return of at least r times the market. It can be shown that this quantity is given by the number $\mathbf{T}(r) = \inf \big\{ T > 0 \,|\, f(T) \leq 1/r \big\}$, the inverse of the decreasing function $f(\cdot)$ of (6.13) evaluated at $1/r$. Details can be found in [FK] (2005).

Question: Can the counterparts of (6.14), (6.15) be computed when one is not allowed to use general strategies $h(\cdot) \in \mathcal{H}(1;T)$, but rather long-only portfolios $\pi(\cdot)$?

6.3 Proof of (6.12)

First, some notation: we shall take $w = 1$ for simplicity, then employ the usual notation $V^\rho(\cdot) \equiv V^{1,\rho}(\cdot)$, $\widehat{V}^\rho(\cdot) \equiv \widehat{V}^{1,\rho}(\cdot)$ for the wealth and the deflated wealth of our given portfolio $\rho(\cdot)$. With

$$h(\cdot) := V^\rho(\cdot)\rho(\cdot) \qquad \text{and} \qquad \theta^\rho(\cdot) := \sigma'(\cdot)\rho(\cdot) - \theta(\cdot)\,,$$

we notice that the equation (6.10) takes the form $d\widehat{V}^\rho(t) = \widehat{V}^\rho(t)\big(\theta^\rho(t)\big)'\, dW(t)$, or equivalently

$$\widehat{V}^\rho(t) = \exp\Big\{ \int_0^t \big(\theta^\rho(s)\big)'\, dW(s) - \frac{1}{2}\int_0^t \|\theta^\rho(s)\|^2\, ds \Big\}. \tag{6.16}$$

On the other hand, introducing the process

$$\widetilde{W}(t) \,:=\, W(t) - \int_0^t \theta^\rho(s) ds \,=\, \widehat{W}(t) - \int_0^t \sigma'(s)\rho(s)\, ds\,, \qquad 0 \leq t \leq T\,, \tag{6.17}$$

we obtain

$$\big(\widehat{V}^\rho(t)\big)^{-1} \,=\, \exp\Big\{ -\int_0^t \big(\theta^\rho(s)\big)'\, d\widetilde{W}(s) - \frac{1}{2}\int_0^t \|\theta^\rho(s)\|^2 ds \Big\}. \tag{6.18}$$

We shall argue (6.12) by contradiction: let us assume that it fails, namely, that $\widehat{V}^\rho(\cdot)$ is a martingale. ¿From the Girsanov theorem, the process $\widetilde{W}(\cdot)$ of (6.17) is then a Brownian motion under the equivalent probability measure $\widetilde{\mathbb{P}}_T(A) := \mathbb{E}\big(\widehat{V}^\rho(T)\, 1_A\big)$ on $\mathcal{F}(T)$. On the other hand, Itô's rule gives

$$d\left(\frac{V^\pi(t)}{V^\rho(t)}\right) \;=\; \left(\frac{V^\pi(t)}{V^\rho(t)}\right) \cdot \sum_{i=1}^n \sum_{\nu=1}^d \big(\pi_i(t) - \rho_i(t)\big)\sigma_{i\nu}(t)\, d\widetilde{W}_\nu(t) \tag{6.19}$$

for any portfolio $\pi(\cdot)$, in conjunction with (6.7), (6.18) and (6.17). Then the uniform boundedness condition (1.14) implies that the ratio $V^\pi(\cdot)/V^\rho(\cdot)$ is a martingale under $\widetilde{\mathbb{P}}_T$; in particular, $\mathbb{E}^{\widetilde{\mathbb{P}}_T}\big(V^\pi(T)/V^\rho(T)\big) = 1$.

Now consider any portfolio $\pi(\cdot)$ that satisfies the conditions of (6.1) on the time-horizon $[0,T]$; such a portfolio exists, because we are operating in the setting of Proposition 6.3. This gives, in particular, $\widetilde{\mathbb{P}}_T\left(V^\pi(T) \geq V^\rho(T)\right) = 1$. In conjunction with the equality $\mathbb{E}^{\widetilde{\mathbb{P}}_T}\left(V^\pi(T)/V^\rho(T)\right) = 1$ just proved, we obtain $\widetilde{\mathbb{P}}_T\left(V^\pi(T) = V^\rho(T)\right) = 1$, or equivalently:

$$\mathbb{P}\big(V^\pi(T) = V^\rho(T)\big) = 1 \qquad \text{for every portfolio } \pi(\cdot) \text{ that satisfies (6.1)}.$$

But this contradicts the second condition $\mathbb{P}\big(V^\pi(T) > V^\rho(T)\big) > 0$ of (6.1).

7 Diversity leads to Arbitrage

We provide now examples which demonstrate the following principle: *If the model $\mathcal{M}$ of (1.1), (1.2) is weakly diverse over the time-interval $[0,T]$, and if (3.10) holds, then $\mathcal{M}$ contains arbitrage opportunities relative to the market portfolio, at least for sufficiently long time horizons $T \in (0,\infty)$.*

The first such examples involve heavily the *diversity-weighted portfolio* $\mu^{(p)}(\cdot) = \big(\mu_1^{(p)}(\cdot), \ldots, \mu_n^{(p)}(\cdot)\big)'$ defined in terms of the market portfolio $\mu(\cdot)$ of (2.1) by

$$\mu_i^{(p)}(t) := \frac{\big(\mu_i(t)\big)^p}{\sum_{j=1}^n \big(\mu_j(t)\big)^p}, \qquad \forall \;\; i = 1, \ldots, n \tag{7.1}$$

for some arbitrary but fixed $p \in (0,1)$. Compared to $\mu(\cdot)$, the portfolio $\mu^{(p)}(\cdot)$ in (7.1) decreases the proportion(s) held in the largest stock(s) and increases those placed in the smallest stock(s), while preserving the relative rankings of all stocks; see (7.8) below. The actual performance of this portfolio relative to the S&P 500 index over a 33-year period is discussed in detail in Fernholz (2002), Chapter 7.

We show below that if the model $\mathcal{M}$ is weakly diverse on a finite time horizon $[0,T]$, then the value process $V^{\mu^{(p)}}(\cdot)$ of the diversity-weighted portfolio in (7.1) satisfies

$$V^{\mu^{(p)}}(T) > V^\mu(T)\big(n^{-1/p}\, e^{\varepsilon\delta T/2}\big)^{1-p} \tag{7.2}$$

almost surely. In particular,

$$\mathbb{P}\left(V^{\mu^{(p)}}(T) > V^\mu(T)\right) = 1, \quad \text{provided that} \quad T \geq \frac{2}{p\varepsilon\delta}\log n, \tag{7.3}$$

and $\mu^{(p)}(\cdot)$ is an arbitrage opportunity relative to the market $\mu(\cdot)$, in the sense of (6.3)-(6.1). The significance of such a result for practical long-term portfolio management cannot be overstated.

What conditions on the coëfficients $b(\cdot)$, $\sigma(\cdot)$ of $\mathcal{M}$ are sufficient for guaranteeing diversity, as in (5.1), over the time horizon $[0,T]$? For simplicity, assume that (3.10) and (1.14) both hold. Then certainly $\mathcal{M}$ cannot be diverse if $b_1(\cdot) - r(\cdot), \ldots, b_n(\cdot) - r(\cdot)$ are bounded uniformly in (t,ω), or even if they satisfy a condition of the Novikov type

$$\mathbb{E}\bigg(\exp\Big\{\frac{1}{2}\int_0^T \big\|b(t) - r(t)\mathbf{I}\big\|^2 dt\Big\}\bigg) < \infty. \tag{7.4}$$

The reason is that, under all these conditions (3.10), (1.14) and (7.4), the process

$$\theta(\cdot) = \sigma'(\cdot)\big(\sigma(\cdot)\sigma(\cdot)\big)^{-1}[\,b(\cdot) - r(\cdot)\mathbf{I}\,]$$

satisfies the requirements (6.4), and the resulting exponential local martingale $Z(\cdot)$ of (6.5) is a true martingale – contradicting Proposition 6.3, at least for sufficiently large $T > 0$.

Proof of (7.3): Let us start by introducing the function

$$\mathbf{G}_p(x) := \Big(\sum_{i=1}^n x_i^p\Big)^{1/p}, \qquad x \in \Delta_+^n, \tag{7.5}$$

which we shall interpret as a "measure of diversity"; see below. An application of Itô's rule to the process $\{\mathbf{G}_p(\mu(t)),\ 0 \le t < \infty\}$ leads after some computation, and in conjunction with (3.9) and the numéraire-invariance property (3.5), to the expression

$$\log\left(\frac{V^{\mu^{(p)}}(T)}{V^{\mu}(T)}\right) = \log\left(\frac{\mathbf{G}_p(\mu(T))}{\mathbf{G}_p(\mu(0))}\right) + (1-p)\int_0^T \gamma^*_{\mu^{(p)}}(t)\,dt \tag{7.6}$$

for the wealth $V^{\mu^{(p)}}(\cdot)$ of the diversity-weighted portfolio $\mu^{(p)}(\cdot)$ of (7.1); see also Section 11 below, particularly (11.2) and its proof in subsection 11.3. One big advantage of the expression (7.6) is that it is free of stochastic integrals, and thus lends itself to pathwise (almost sure) comparisons.

For the function of (7.5), we have the simple bounds

$$1 = \sum_{i=1}^n \mu_i(t) \le \sum_{i=1}^n \big(\mu_i(t)\big)^p = \big(\mathbf{G}_p(\mu(t))\big)^p \le n^{1-p}$$

(minimum diversity occurs when the entire market is concentrated in one stock, and maximum diversity when all stocks have the same capitalization), so that the function of (7.5) satisfies

$$\log\left(\frac{\mathbf{G}_p(\mu(T))}{\mathbf{G}_p(\mu(0))}\right) \ge -\frac{1-p}{p}\log n. \tag{7.7}$$

This shows that $V^{\mu^{(p)}}(\cdot)/V^{\mu}(\cdot)$ is bounded from below by the constant $m^{-(1-p)/p}$, so (6.3) is satisfied for $\rho(\cdot) \equiv \mu(\cdot)$ and $\pi(\cdot) \equiv \mu^{(p)}(\cdot)$.

On the other hand, we have already remarked that the biggest weight of the portfolio $\mu^{(p)}(\cdot)$ in (7.1) does not exceed the largest market weight:

$$\mu^{(p)}_{(1)}(t) := \max_{1\le i\le n} \mu^{(p)}_i(t) = \frac{\big(\mu_{(1)}(t)\big)^p}{\sum_{k=1}^n \big(\mu_{(k)}(t)\big)^p} \le \mu_{(1)}(t)\,. \tag{7.8}$$

(The reverse inequality holds for the smallest weights: $\mu^{(p)}_{(n)}(t) := \min_{1\le i\le n} \mu^{(p)}_i(t) \ge \mu_{(n)}(t)$.)

We have assumed that the market is weakly diverse over $[0,T]$, namely, that there is some $0<\delta<1$ for which $\int_0^T \big(1-\mu_{(1)}(t)\big)dt > \delta T$ holds almost surely. From (3.12) and (7.8), this implies

$$\int_0^T \gamma^*_{\mu^{(p)}}(t)\,dt \ge \frac{\varepsilon}{2}\int_0^T \Big(1-\mu^{(p)}_{(1)}(t)\Big)\,dt \ge \frac{\varepsilon}{2}\int_0^T \big(1-\mu_{(1)}(t)\big)\,dt > \frac{\varepsilon}{2}\delta T$$

a.s. In conjunction with (7.7), this leads to (7.2) and (7.3) via

$$\log\left(\frac{V^{\mu^{(p)}}(T)}{V^{\mu}(T)}\right) > (1-p)\Big(\frac{\varepsilon T}{2}\delta - \frac{1}{p}\log n\Big). \qquad \Box \tag{7.9}$$

If $\mathcal{M}$ is uniformly weakly diverse and strongly non-degenerate over an interval $[T_0,\infty)$, then (7.9) implies that the market portfolio will lag rather significantly behind the diversity-weighted portfolio over long time horizons. To wit, that (6.2) will hold:

$$\mathcal{L}^{\mu^{(p)},\mu} = \liminf_{T\to\infty} \frac{1}{T}\log\Big(V^{\mu^{(p)}}(T)/V^{\mu}(T)\Big) \ge (1-p)\varepsilon\delta/2 > 0, \qquad \text{a.s.}$$

In Figure 1 we see the cumulative changes in the diversity of the U.S. stock market over the period from 1927 to 2004, measured by $\mathbf{G}_p(\cdot)$ with $p = 1/2$. The chart shows the cumulative changes in diversity due to capital gains and losses, rather than absolute diversity, which is affected by changes in market composition and corporate actions. Considering only capital gains and losses

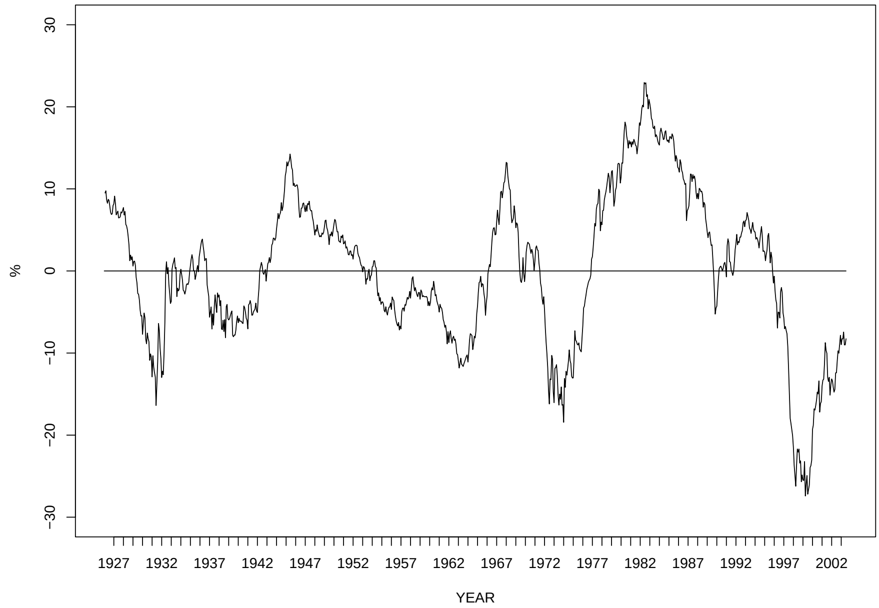

Figure 1: Cumulative change in market diversity, 1927–2004.

has the same effect as adjusting the "divisor" of an equity index. The values used in Figure 1 have been normalized so that the average over the whole period is zero. We can observe from the chart that diversity appears to be mean-reverting over the long term, with intermediate trends of 10 to 20 years. The extreme lows for diversity seem to accompany bubbles: the Great Depression, the "nifty fifty" era, and the "irrational exuberance" period.

7.1 Remark. (Fernholz, 2002): Under the conditions of this section, consider the portfolio with weights

$$\pi_i(t) = \left(\frac{2 - \mu_i(t)}{\mathbf{G}(\mu(t))} - 1 \right) \mu_i(t), \quad 1 \le i \le n, \quad \text{where} \quad \mathbf{G}(x) := 1 - \frac{1}{2} \sum_{i=1}^{n} x_i^2$$

for $x \in \Delta^n$. It can be shown that this portfolio leads to arbitrage relative to the market, over sufficiently long time horizons $[0, T]$, namely with $T \ge (2n/\varepsilon\delta^2) \log 2$. In this case, we also have $\pi_i(t) \le 3\mu_i(t)$, for all $t \in [0, T]$, a.s., so, with appropriate initial conditions, there is no risk that this $\pi(\cdot)$ will hold more of a stock than the market holds.

8 Mirror Portfolios, Short-Horizon Arbitrage

In the previous section we saw that in weakly diverse markets which satisfy the strict non-degeneracy condition (3.10), one can construct explicitly simple portfolios that lead to arbitrages relative to the market over sufficiently long time horizons. The purpose of this section is to demonstrate that, under these same conditions, such arbitrages exist indeed over *arbitrary* time horizons, no matter how small.

For any given portfolio $\pi(\cdot)$ and real number $q \neq 0$, define the *q-mirror image of* $\pi(\cdot)$ *with respect to the market portfolio*, as

$$\widetilde{\pi}^{[q]}(\cdot) := q\pi(\cdot) + (1 - q)\mu(\cdot).$$

This is clearly a portfolio; and it is long-only if $\pi(\cdot)$ itself is long-only and $0 < q < 1$. If $q = -1$, we call $\widetilde{\pi}^{[-1]}(\cdot) = 2\mu(\cdot) - \pi(\cdot)$ the "mirror image" of $\pi(\cdot)$ with respect to the market.

By analogy with (2.6), let us define the *relative covariance of* $\pi(\cdot)$ *with respect to the market*, as

$$\tau^{\pi}_{\mu\mu}(t) := \big(\pi(t)-\mu(t)\big)' a(t) \big(\pi(t)-\mu(t)\big), \qquad 0 \le t \le T.$$

8.1 Remark. Recall from (2.8) the fact $\tau^{\mu}(t)\mu(t) \equiv 0$, and establish the elementary properties $\tau^{\pi}_{\mu\mu}(t) = \pi'(t)\tau^{\mu}(t)\pi(t) = \tau^{\mu}_{\pi\pi}(t)$ and $\tau^{\mu}_{\widetilde{\pi}^{[q]}\widetilde{\pi}^{[q]}}(t) = q^2 \tau^{\mu}_{\pi\pi}(t)$.

8.2 Remark. The wealth of $\widetilde{\pi}^{[q]}(\cdot)$ relative to the market, can be computed as

$$\log\left(\frac{V^{\widetilde{\pi}^{[q]}}(T)}{V^{\mu}(T)}\right) = q \, \log\left(\frac{V^{\pi}(T)}{V^{\mu}(T)}\right) + \frac{q(1-q)}{2}\int_0^T \tau^{\mu}_{\pi\pi}(t)\, dt.$$

Indeed, let us write the second equality in (3.4) with $\pi(\cdot)$ replaced by $\widetilde{\pi}^{[q]}(\cdot)$, and recall $\widetilde{\pi}^{[q]} - \mu = q(\pi-\mu)$. From the resulting expression, let us subtract the second equality in (3.4), now multiplied by q ; the result is

$$\frac{d}{dt}\left(\log\frac{V^{\widetilde{\pi}^{[q]}}(t)}{V^{\mu}(t)} - q\,\log\frac{V^{\pi}(t)}{V^{\mu}(t)}\right) = (q-1)\gamma^*_{\mu}(t) + \big(\gamma^*_{\widetilde{\pi}^{[q]}}(t) - q\gamma^*_{\mu}(t)\big).$$

But from the equalities of Remark 8.1 and Lemma 3.3, we obtain

$$\begin{aligned} 2\big(\gamma^*_{\widetilde{\pi}^{[q]}}(t) - q\gamma^*_{\pi}(t)\big) &= \sum_{i=1}^{n} \big(\widetilde{\pi}^{[q]}(t) - q\pi_i(t)\big)\tau^{\mu}_{ii}(t) - \tau^{\mu}_{\widetilde{\pi}^{[q]}\widetilde{\pi}^{[q]}}(t) + q\tau^{\mu}_{\pi\pi}(t) \\ &= (1-q)\sum_{i=1}^{n} \mu_i(t)\tau^{\mu}_{ii}(t) + q\tau^{\mu}_{\pi\pi}(t) - q^2\tau^{\mu}_{\pi\pi}(t) = (1-q)\big(2\gamma^*_{\mu}(t) + q\tau^{\mu}_{\pi\pi}(t)\big). \end{aligned}$$

The desired equality now follows.

8.3 Remark. Suppose that the portfolio $\pi(\cdot)$ satisfies

$$\mathbb{P}\big(V^{\pi}(T)/V^{\mu}(T) \ge \beta\big) = 1 \qquad \text{or} \qquad \mathbb{P}\big(V^{\pi}(T)/V^{\mu}(T) \le 1/\beta\big) = 1$$

and

$$\mathbb{P}\Big(\int_0^T \tau^{\mu}_{\pi\pi}(t)\, dt \ge \eta\Big) = 1$$

for some real numbers $T > 0$, $\eta > 0$ and $0 < \beta < 1$. Then there exists another portfolio $\widehat{\pi}(\cdot)$ with $\mathbb{P}\big(V^{\widehat{\pi}}(T) < V^{\mu}(T)\big) = 1$.

To see this, suppose first that we have $\mathbb{P}\big(V^{\pi}(T)/V^{\mu}(T) \le 1/\beta\big) = 1$; then we can just take $\widehat{\pi}(\cdot) \equiv \widetilde{\pi}^{[q]}(\cdot)$ with $q > 1 + (2/\eta)\log(1/\beta)$, for then Remark 8.2 gives

$$\log\left(\frac{V^{\widetilde{\pi}^{[q]}}(T)}{V^{\mu}(T)}\right) \le q\left(\log\big(1/\beta\big) + \frac{1-q}{2}\eta\right) < 0, \qquad \text{a.s.}$$

If, on the other hand, $\mathbb{P}\big(V^{\pi}(T)/V^{\mu}(T) \ge \beta\big) = 1$ holds, then similar reasoning shows that it suffices to take $\widehat{\pi}(\cdot) \equiv \widetilde{\pi}^{[q]}(\cdot)$ with $q \in \big(0, 1-(2/\eta)\log(1/\beta)\big)$.

8.1 A "Seed" Portfolio

Now let us consider $\pi = e_1 = (1, 0, \cdots, 0)'$ and the market portfolio $\mu(\cdot)$; we shall fix a real number $q > 1$ in a moment, and define the portfolio

$$\widehat{\pi}(t) := \widetilde{\pi}^{[q]}(t) = qe_1 + (1-q)\mu(t), \qquad 0 \le t < \infty \tag{8.1}$$

which takes a long position in the first stock and a short position in the market. In particular, $\widehat{\pi}_1(t) = q + (1-q)\mu_1(t)$ and $\widehat{\pi}_i(t) = (1-q)\mu_i(t)$ for $i = 2, \cdots, n$. Then we have

$$\log\left(\frac{V^{\widehat{\pi}}(T)}{V^{\mu}(T)}\right) = q\,\log\left(\frac{\mu_1(T)}{\mu_1(0)}\right) - \frac{q(q-1)}{2}\int_0^T \tau_{11}^{\mu}(t)dt \tag{8.2}$$

from Remark 8.2. But taking $\beta := \mu_1(0)$ we have $(\mu_1(T)/\mu_1(0)) \le 1/\beta$; and if the market is weakly diverse on $[0,T]$ and satisfies the strict non-degeneracy condition (3.10), we obtain from (3.11) and the Cauchy-Schwarz inequality

$$\int_0^T \tau_{11}^{\mu}(t)dt \ge \varepsilon \int_0^T \left(1-\mu_{(1)}\right)^2 dt > \varepsilon\delta^2 T =: \eta\,. \tag{8.3}$$

Recalling Remark 8.3, we see that *the market portfolio represents then an arbitrage opportunity with respect to the portfolio* $\widehat{\pi}(\cdot)$ *of (8.1)*, provided that for any given $T \in (0,\infty)$ we select

$$q > q(T) := 1 + (2/\varepsilon\delta^2 T)\log\left(1/\mu_1(0)\right). \tag{8.4}$$

The portfolio $\widehat{\pi}(\cdot)$ of (8.1) can be used as a "seed", to create *long-only* portfolios that outperform the market portfolio $\mu(\cdot)$, *over any given time horizon* $T \in (0,\infty)$. The idea is to immerse $\widehat{\pi}(\cdot)$ in a sea of market portfolio, swamping the short positions while retaining the essential portfolio characteristics. Crucial in these constructions is the a.s. comparison, a consequence of (8.2):

$$V^{\widehat{\pi}}(t) \le \left(\frac{\mu_1(t)}{\mu_1(0)}\right)^q V^{\mu}(t)\,, \qquad 0 \le t < \infty\,. \tag{8.5}$$

8.2 Relative Arbitrage on Arbitrary Time Horizons

To implement this idea, consider a strategy $h(\cdot)$ that, at time $t=0$, invests $q/(\mu_1(0))^q$ dollars in the market portfolio, goes one dollar short in the portfolio $\widehat{\pi}(\cdot)$ of (8.1), and makes no change thereafter. The number $q > 1$ is chosen again as in (8.4). The wealth generated by this strategy, with initial capital $z := q/(\mu_1(0))^q - 1 > 0$, is

$$\mathcal{V}^{z,h}(t) = \frac{qV^{\mu}(t)}{(\mu_1(0))^q} - V^{\widehat{\pi}}(t) \ge \frac{V^{\mu}(t)}{(\mu_1(0))^q}\left(q - (\mu_1(t))^q\,\right) > 0, \qquad 0 \le t < \infty, \tag{8.6}$$

thanks to (8.5) and $q > 1 > (\mu_1(t))^q$. This process $\mathcal{V}^{z,h}(\cdot)$ coincides with the wealth $V^{z,\eta}(\cdot)$ generated by a portfolio $\eta(\cdot)$ with weights

$$\eta_i(t) = \frac{1}{\mathcal{V}^{z,h}(t)}\left(\frac{q\mu_i(t)}{(\mu_1(0))^q}V^{\mu}(t) - \widehat{\pi}_i(t)V^{\widehat{\pi}}(t)\right), \qquad i = 1, \cdots, n \tag{8.7}$$

that satisfy $\sum_{i=1}^n \eta_i(t) = 1$. Now we have $\widehat{\pi}_i(t) = -(q-1)\mu_i(t) < 0$ for $i = 2, \cdots, n$, so the quantities $\eta_2(\cdot), \ldots, \eta_n(\cdot)$ are strictly positive. To check that $\eta(\cdot)$ is a long-only portfolio, we have to verify $\eta_1(t) \ge 0$; but the dollar amount invested by $\eta(\cdot)$ in the first stock at time t, namely

$$\frac{q\mu_1(t)}{(\mu_1(0))^q}\,V^{\mu}(t) - \left[q - (q-1)\mu_1(t)\right]V^{\widehat{\pi}}(t)$$

dominates $\frac{q\mu_1(t)}{(\mu_1(0))^q}V^{\mu}(t) - \left[q-(q-1)\mu_1(t)\right]\left(\frac{\mu_1(t)}{\mu_1(0)}\right)^q V^{\mu}(t)$, or equivalently

$$\frac{V^{\mu}(t)\mu_1(t)}{(\mu_1(0))^q}\Big((q-1)(\mu_1(t))^q + q\left[1-(\mu_1(t))^{q-1}\right]\Big) > 0\,,$$

again thanks to (8.5) and $q > 1 > (\mu_1(t))^{q-1}$. Thus $\eta(\cdot)$ is indeed a long-only portfolio.

On the other hand, $\eta(\cdot)$ *outperforms at* $t = T$ *a market portfolio that starts with the same initial capital*; this is because $\eta(\cdot)$ is long in the market $\mu(\cdot)$ and short in the portfolio $\widehat{\pi}(\cdot)$, which underperforms the market at $t = T$. Indeed, from Remark 8.3 we have

$$V^{z,\eta}(T) \;=\; \frac{q}{(\mu_1(0))^q} V^{\mu}(T) - V^{\widehat{\pi}}(T) > zV^{\mu}(T) = V^{z,\mu}(T)\,, \quad \text{a.s.}$$

Note, however, that as $T \downarrow 0$, the initial capital $z(T) = q(T)/(\mu_1(0))^{q(T)} - 1$ required to do all of this, increases without bound: It may take a huge amount of initial investment to realize the extra basis point's worth of relative arbitrage over a short time horizon — confirming of course, if confirmation is needed, that *time is money...*

9 A Diverse Market Model

The careful reader might have been wondering whether the theory we have developed so far may turn out to be vacuous. *Do there exist market models of the form* (1.1), (1.2) *that are diverse, at least weakly?* This is of course a very legitimate question.

Let us mention then, rather briefly, an example of such a market model $\mathcal{M}$ which is diverse over any given time horizon $[0,T]$ with $0 < T < \infty$, and indeed satisfies the conditions of subsection 4.1 as well. For the details of this construction we refer to [FKK] (2005).

With given $\delta \in (1/2, 1)$, equal numbers of stocks and driving Brownian motions (that is, $d = n$), constant volatility matrix σ that satisfies (3.10), and non-negative numbers $g_1, \ldots, g_n$, we take a model

$$d \log X_i(t) = \gamma_i(t)\, dt + \sum_{\nu=1}^{n} \sigma_{i\nu}\, dW_\nu(t), \qquad 0 \le t \le T \tag{9.1}$$

in the form (1.5) for the vector $\mathfrak{X}(\cdot) = \big(X_1(\cdot), \cdots, X_n(\cdot)\big)'$ of stock prices. With the usual notation $X(t) = \sum_{j=1}^{n} X_j(t)$, its growth rates are specified as

$$\gamma_i(t) := g_i 1_{\mathcal{Q}_i^c}(\mathfrak{X}(t)) - \frac{M}{\delta}\, \frac{1_{\mathcal{Q}_i}(\mathfrak{X}(t))}{\log\big((1-\delta)X(t)/X_i(t)\big)}. \tag{9.2}$$

In other words, $\gamma_i(t) = g_i \ge 0$ if $\mathfrak{X}(t) \notin \mathcal{Q}_i$ (the i^{th} stock does not have the largest capitalization); and

$$\gamma_i(t) = -\frac{M}{\delta}\, \frac{1}{\log\big((1-\delta)/\mu_i(t)\big)}, \qquad \text{if} \quad \mathfrak{X}(t) \in \mathcal{Q}_i \tag{9.3}$$

(the i^{th} stock *does* have the largest capitalization). We are setting here

$$\mathcal{Q}_1 := \Big\{ x \in (0,\infty)^n \,\big|\, x_1 \ge \max_{2 \le j \le n} x_j \Big\}, \quad \mathcal{Q}_n := \Big\{ x \in (0,\infty)^n \,\big|\, x_n > \max_{1 \le j \le m-1} x_j \Big\},$$
$$\mathcal{Q}_i := \Big\{ x \in (0,\infty)^n \,\big|\, x_i > \max_{1 \le j \le i-1} x_j,\ x_i \ge \max_{i+1 \le j \le n} x_j \Big\} \quad \text{for} \;\; i = 2, \ldots, n-1.$$

With this specification (9.2), (9.3), all stocks but the largest behave like geometric Brownian motions (with growth rates $g_i \ge 0$ and variances $a_{ii} = \sum_{\nu=1}^{n} \sigma_{i\nu}^2$), whereas the log-price of the largest stock is subjected to a log-pole-type singularity in its drift, away from an appropriate right boundary.

One can then show that the resulting system of stochastic differential equations has a unique, strong solution (so the filtration $\mathbb{F}$ is now the one generated by the driving $n-$dimensional Brownian motion), and that the diversity requirement (5.1) is satisfied on any given time horizon. Such models can be modified appropriately, to create ones that are weakly diverse but not diverse; see [FK] (2005) for details.

Slightly more generally, in order to guarantee diversity it is enough to require

$$\min_{2 \le k \le n} \gamma_{(k)}(t) \ge 0 \ge \gamma_{(1)}(t), \qquad \min_{2 \le k \le n} \gamma_{(k)}(t) - \gamma_{(1)}(t) + \frac{\varepsilon}{2} \ge \frac{M}{\delta} F(Q(t)),$$

where $Q(t) := \log\big((1-\delta)/\mu_{(1)}(t)\big)$.

Here the function $F : (0,\infty) \to (0,\infty)$ is taken to be continuous, and such that the associated scale function

$$U(x) := \int_1^x \exp\Big\{-\int_1^y F(z)\,dz\Big\}dy, \quad x \in (0,\infty) \qquad \text{satisfies} \quad U(0+) = -\infty;$$

for instance, we have $U(x) = \log x$ when $F(x) = 1/x$ as above. Under these conditions, it can then be shown that the process $Q(\cdot)$ satisfies $\int_0^T (Q(t))^{-2}dt < \infty$ a.s., and this leads to the a.s. square-integrability

$$\sum_{i=1}^n \int_0^T (b_i(t))^2 dt < \infty$$

of the induced rates of return of the individual stocks

$$b_i(t) = \frac{1}{2}a_{ii} + g_i 1_{\mathcal{Q}_i^c}(\mathfrak{X}(t)) - \frac{M}{\delta}\frac{1_{\mathcal{Q}_i}(\mathfrak{X}(t))}{\log\big((1-\delta)X(t)/X_i(t)\big)}\,, \qquad i = 1, \cdots, n.$$

This square-integrability property is, of course, crucial: it guarantees that the market-price-of-risk process $\theta(\cdot) := \sigma^{-1}b(\cdot)$ is square-integrable a.s., so the exponential local martingale $Z(\cdot)$ of (6.5) is well defined (we are assuming $r(\cdot) \equiv 0$ in all this). Thus the results of Propositions 6.3, 6.5 and Remark 6.4 are applicable to this model.

For additional examples, and for an interesting probabilistic construction that leads to arbitrage, see Osterrieder & Rheinländer (2006).

10 Hedging and Optimization without EMM

Let us broach now the issue of hedging contingent claims in a market such as that of subsection 6.1, and over a time horizon $[0,T]$ for which (6.1) is satisfied.

Consider first a *European contingent claim*, that is, an $\mathcal{F}(T)-$measurable random variable $Y : \Omega \to [0,\infty)$ with

$$0 < y := \mathbb{E}\big(YZ(T)/B(T)\big) < \infty \tag{10.1}$$

in the notation of (6.5). From the point of view of the seller of the contingent claim (e.g., stock option), this random amount represents a liability that has to be covered with the right amount of initial funds at time $t = 0$ and the right trading strategy during the interval $[0,T]$, so that at the end of the period (time $t = T$) the initial funds have grown enough, to cover the liability without risk. Thus, the seller is interested in the so-called *upper hedging price*

$$\mathcal{U}^Y(T) := \inf\big\{w > 0\,|\,\exists\, h(\cdot) \in \mathcal{H}(w;T) \;\text{ s.t. } \mathcal{V}^{w,h}(T) \geq Y, \text{ a.s.}\big\}, \tag{10.2}$$

the smallest amount of initial capital that makes such riskless hedging possible.

The standard theory of mathematical finance assumes that $\mathfrak{M}$, the set of equivalent martingale measures for the model $\mathcal{M}$, is non-empty; then shows that $\mathcal{U}^Y(T)$ can be computed as

$$\mathcal{U}^Y(T) \;=\; \sup_{\mathbb{Q}\in\mathfrak{M}} \mathbb{E}^{\mathbb{Q}}\big(Y/B(T)\big), \tag{10.3}$$

the supremum of the claim's discounted expected value over this set of probability measures. In our context no EMM exists (that is, $\mathfrak{M} = \emptyset$), so the approach breaks down and the problem seems hopeless.

Not quite, though: there is still a long way one can go, simply by utilizing the availability of the strict local martingale $Z(\cdot)$ (and of the associated "deflator" $Z(\cdot)/B(\cdot)$), as well as the properties (6.9), (6.10) of the processes in (6.8). For instance, if the set on the right-hand side of (10.2) is not empty, then for any $w > 0$ in this set and for any $h(\cdot) \in \mathcal{H}(w;T)$, the local martingale $\widehat{\mathcal{V}}^{w,h}(\cdot)$ of (6.8) is non-negative, thus a supermartingale. This gives

$$w \geq \mathbb{E}\big(\mathcal{V}^{w,h}(T)Z(T)/B(T)\big) \geq \mathbb{E}\big(YZ(T)/B(T)\big) = y,$$

and because $w > 0$ is arbitrary we deduce the inequality $\mathcal{U}^Y(T) \geq y$ (which holds trivially if the set of (10.2) is empty, since then $\mathcal{U}^Y(T) = \infty$).

10.1 Completeness without an EMM

To obtain the reverse inequality we shall assume that $n = d$, i.e., that we have exactly as many sources of randomness as there are stocks in the market $\mathcal{M}$, and that the filtration $\mathbb{F}$ is generated by the driving Brownian Motion $W(\cdot)$ in (1.1): $\mathbb{F} = \mathbb{F}^W$.

With these assumptions, one can represent the non-negative martingale $M(t) := \mathbb{E}\big(YZ(T)/B(T)|\mathcal{F}(t)\big)$, $0 \leq t \leq T$ as a stochastic integral

$$M(t) = y + \int_0^t \psi'(s)dW(s) \geq 0, \qquad 0 \leq t \leq T \tag{10.4}$$

for some progressively measurable and a.s. square-integrable process $\psi : [0,T] \times \Omega \to \mathbb{R}^d$. Setting $V_*(\cdot) := M(\cdot)B(\cdot)/Z(\cdot)$ and $h_*(\cdot) := (B(\cdot)/Z(\cdot))a^{-1}(\cdot)\ \sigma(\cdot)\big(\psi(\cdot) + V_*(\cdot)\theta(\cdot)\big)$, then comparing (6.10) with (10.4), we observe $V_*(0) = y$, $V_*(T) = Y$ and $V_*(\cdot) \equiv \mathcal{V}^{y,h_*}(\cdot) \geq 0$, almost surely.

Therefore, the trading strategy $h_*(\cdot)$ is in $\mathcal{H}(y;T)$ and satisfies the *exact replication property* $\mathcal{V}^{y,h_*}(T) = Y$ a.s. This implies that y belongs to the set on the right-hand side of (10.2), and so $y \geq \mathcal{U}^Y(T)$. But we have already established the reverse inequality, actually in much greater generality, so recalling (10.1) we get the *Black-Scholes-type formula*

$$\mathcal{U}^Y(T) = \mathbb{E}\big(YZ(T)/B(T)\big) \tag{10.5}$$

for the upper hedging price of (10.2), under the assumptions of the first paragraph in this subsection. In particular, we see that a market $\mathcal{M}$ which is weakly diverse – hence without an equivalent probability measure under which discounted stock prices are (at least local) martingales – can nevertheless be *complete*. Similar observations have been made by Lowenstein & Willard (2000.a,b) and by Platen (2002, 2006).

10.2 Ramifications and Open Problems

10.1 Example. A European Call Option. Consider the contingent claim $Y = \big(X_1(T) - q\big)^+$: this is a European call-option with strike $q > 0$ on the first stock. Let us assume also that the interest-rate process $r(\cdot)$ is bounded away from zero, namely that $\mathbb{P}(r(t) \geq r,\ \forall t \geq 0) = 1$ holds for some $r > 0$, and that the market $\mathcal{M}$ is weakly diverse on all sufficiently large time horizons $T \in (0,\infty)$. Then for the hedging price of this contingent claim we have from Remark 6.4, (10.5), Jensen's inequality, and $\mathbb{E}(Z(T)) < 1$:

$$\begin{aligned} X_1(0) &> \mathbb{E}\big(Z(T)X_1(T)/B(T)\big) \geq \mathbb{E}\big(Z(T)(X_1(T)-q)^+/B(T)\big) = \mathcal{U}^Y(T) \\ &\geq \Big(\mathbb{E}\big(Z(T)X_1(T)/B(T)\big) - q\,\mathbb{E}\big(Z(T)e^{-\int_0^T r(t)dt}\big)\Big)^+ \\ &\geq \Big(\mathbb{E}\big(Z(T)X_1(T)/B(T)\big) - q\,e^{-rT}\mathbb{E}Z(T)\Big)^+ \\ &\geq \Big(\mathbb{E}\big(Z(T)X_1(T)/B(T)\big) - q\,e^{-rT}\Big)^+, \end{aligned}$$

thus

$$0 \leq \mathcal{U}^Y(\infty) := \lim_{T\to\infty} \mathcal{U}^Y(T) = \lim_{T\to\infty} \downarrow \mathbb{E}\big(Z(T)X_1(T)/B(T)\big) < X_1(0). \tag{10.6}$$

The upper hedging price of the option is *strictly less* than the capitalization of the underlying stock at time $t = 0$, and tends to $\mathcal{U}^Y(\infty) \in [0, X_1(0))$ as the time-horizon increases without limit.

If $\mathcal{M}$ is weakly diverse uniformly over some $[T_0, \infty)$, then the limit in (10.6) is actually zero: *a European call-option that can never be exercised, has hedging price equal to zero.* Indeed, for every fixed $p \in (0,1)$ and $T \geq \frac{2\log n}{p\varepsilon\delta} \vee T_0$, and with the normalization $X(0) = 1$, the quantity

$$\mathbb{E}\bigg(\frac{Z(T)}{B(T)}X_1(T)\bigg) \leq \mathbb{E}\bigg(\frac{Z(T)}{B(T)}V^{\mu}(T)\bigg) \leq \mathbb{E}\bigg(\frac{Z(T)}{B(T)}V^{\mu^{(p)}}(T)\bigg)\, n^{\frac{1-p}{p}}e^{-\varepsilon\delta(1-p)T/2}$$

is dominated by $n^{\frac{1-p}{p}}e^{-\varepsilon\delta(1-p)T/2}$, from (7.2), (2.2) and the supermartingale property of the process $Z(\cdot)V^{\mu^{(p)}}(\cdot)/B(\cdot)$. Letting $T \to \infty$ we obtain $\mathcal{U}^Y(\infty) = 0$.

10.2 Remark. Note the sharp difference between this case and the situation where an equivalent martingale measure exists on every finite time horizon; namely, when both $Z(\cdot)$ and $Z(\cdot)X_1(\cdot)/B(\cdot)$ are martingales. Then we have $\mathbb{E}(Z(T)X_1(T)/B(T)) = X_1(0)$ for all $T \in (0,\infty)$, and $\mathcal{U}^Y(\infty) = X_1(0)$: as the time horizon increases without limit, the hedging price of the call option approaches the current stock value (see [KS] (1998), p.62).

10.3 Remark. The above theory extends to the case $d > n$ of *incomplete markets*, and more generally to *closed, convex constraints* on portfolio choice as in Chapter 5 of [KS] (1998), under the conditions of (6.4). See the paper [KK] (2006) for a treatment of these issues in a general semimartingale setting.

In particular, the Black-Scholes-type formula (10.5) can be generalized, in the spirit of (10.3), to the case $d > n$ and filtration $\mathbb{F}$ not necessarily equal to the Brownian filtration $\mathbb{F}^W$. Let Θ be the set of $\mathbb{F}-$progressively measurable processes $\theta(\cdot)$ that satisfy the requirements of (6.4) and, for each $\theta(\cdot) \in \Theta$, denote by $Z_\theta(\cdot)$ the process of (6.5). Then the upper hedging price of (10.2) is given as

$$\mathcal{U}^Y(T) = \sup_{\theta(\cdot)\in\Theta} \mathbb{E}\big(YZ_\theta(T)/B(T)\big)\,. \tag{10.7}$$

10.4 Remark. Open Question: Develop a theory for pricing *American contingent claims* under the assumptions of the present section. As C. Kardaras (2006) observes, in the absence of an EMM it is not optimal to exercise an American call option (written on a non-dividend-paying stock) only at maturity $t = T$. Can one then characterize, or compute, the optimal exercise time?

10.3 Utility Maximization in the Absence of EMM

Suppose we are given initial capital $w > 0$, a finite time horizon $T > 0$, and a utility function $u : (0,\infty) \to \mathbb{R}$ (strictly increasing, strictly concave, of class $\mathcal{C}^1$, with $u'(0) := \lim_{x\downarrow 0} u'(x) = \infty$, $u'(\infty) := \lim_{x\to\infty} u'(x) = 0$ and $u(0) := \lim_{x\downarrow 0} u(x)$). The problem is to compute the maximal expected utility

$$\mathfrak{u}(w) := \sup_{h(\cdot)\in\mathcal{H}(w;T)} \mathbb{E}\big(u\big(\mathcal{V}^{w,h}(T)\big)\big)$$

from terminal wealth; to decide whether the supremum is attained; and if so, to identify a strategy $\widehat{h}(\cdot) \in \mathcal{H}(w;T)$ that attains it. We place ourselves under the assumptions of the present section, including those of subsection 10.1.

10.5 Remark. The solution to this question is given by the replicating strategy $\widehat{h}(\cdot) \in \mathcal{H}_+(w;T)$ for the contingent claim

$$\Upsilon = I\big(\Xi(w)D(T)\big), \qquad \text{where} \quad D(t) := Z(t)/B(t) \quad \text{for} \ \ 0 \leq t \leq T\,,$$

in the sense $\mathcal{V}^{w,\widehat{h}}(T) = \Upsilon$ a.s. Here $I : (0,\infty) \to (0,\infty)$ is the inverse of the strictly decreasing marginal utility function $u' : (0,\infty) \to (0,\infty)$, and $\Xi : (0,\infty) \to (0,\infty)$ the inverse of the strictly decreasing function $\mathcal{W}(\cdot)$ given by

$$\mathcal{W}(\xi) := \mathbb{E}\big[\, D(T)I\left(\xi D(T)\right)\,\big]\,, \quad 0 < \xi < \infty,$$

which we are assuming to be $(0,\infty)-$valued.

In the case of the logarithmic utility function $u(x) = \log x,\ x \in (0,\infty)$, it is easily shown that the "log-optimal" trading strategy $h_*(\cdot) \in \mathcal{H}_+(w;T)$ and its associated wealth process $V_*(\cdot) \equiv \mathcal{V}^{w,h_*}(\cdot)$ are given, respectively, by

$$h_*(t) = V_*(t)a^{-1}(t)\big[\,b(t) - r(t)\mathbf{I}\big], \quad V_*(t) = w/D(t) \tag{10.8}$$

for $0 \le t \le T$. Note also that the discounted log-optimal wealth process satisfies

$$d\big(V_*(t)/B(t)\big) = \big(V_*(t)/B(t)\big)\theta'(t)\big[\,\theta(t)\,dt + dW(t)\big]. \tag{10.9}$$

Note that no assumption is been made regarding the existence of an EMM; $Z(\cdot)$ does not have to be a martingale. See Karatzas, Lehoczky, Shreve & Xu (1991) for more information on this problem and on its much more interesting *incomplete market* version $d > n$, under the assumption that the volatility matrix $\sigma(\cdot)$ is of full (row) rank and without assuming the existence of EMM.

The log-optimal trading strategy of (10.8) has some obviously desirable features, discussed in the next remark. But unlike the functionally generated portfolios of the next section, it needs for its implementation knowledge of the covariance structure and of the mean rates of return; these are hard to estimate in practice.

10.6 Remark. The "Numéraire" Property: Assume that the log-optimal strategy of (10.8) is defined for all $0 \le t < \infty$; it has then the following *numéraire property*

$$\mathcal{V}^{w,h}(\cdot)/\mathcal{V}^{w,h_*}(\cdot) \quad \text{is a supermartingale,} \ \ \forall \ \ h(\cdot) \in \mathcal{H}_+(w), \tag{10.10}$$

and from this, one can derive the *asymptotic growth optimality* property

$$\limsup_{t\to\infty} \frac{1}{t}\log\left(\frac{\mathcal{V}^{w,h}(t)}{\mathcal{V}^{w,h_*}(t)}\right) \le 0 \quad \text{a.s.,} \quad \forall \ \ h(\cdot) \in \mathcal{H}_+(w)\,.$$

For a detailed study of these issues, see [KK] (2006).

10.7 Remark. (Platen 2006): The equation for $\Psi(\cdot) := V_*(\cdot)/B(\cdot) = w/Z(\cdot)$ in (10.9) can be written as

$$d\Psi(t) = \alpha(t)\,dt + \sqrt{\Psi(t)\alpha(t)}\,d\mathfrak{B}(t), \qquad \Psi(0) = w$$

where $\mathfrak{B}(\cdot)$ is one-dimensional Brownian motion, and $\alpha(t) := \Psi(\cdot)\|\theta(\cdot)\|^2$.

Then $\Psi(\cdot)$ is a time-changed and scaled *squared Bessel process in dimension 4* (sum of squares of four independent Brownian motions); that is, $\Psi(\cdot) = \mathfrak{X}\big(A(\cdot)\big)/4$, where

$$A(\cdot) := \int_0^{\cdot} \alpha(s)\,ds \qquad \text{and} \qquad \mathfrak{X}(u) = 4(w+u) + 2\int_0^u \sqrt{\mathfrak{X}(v)}\,d\mathfrak{b}(v), \quad u \ge 0$$

in terms of yet another standard, one-dimensional Brownian motion $\mathfrak{b}(\cdot)$.

Part III

Functionally Generated Portfolios

Functionally generated portfolios were introduced in Fernholz (1999a), and generalize broadly the diversity-weighted portfolios of Section 7. For this new class of portfolios one can derive a decomposition of their relative return analogous to that of (7.6), and this proves useful in the construction and study of arbitrages relative to the market. Just as in (7.6), this new decomposition (11.2) does not involve stochastic integrals, and opens the possibility for making probability-one comparisons over given, fixed time-horizons.

11 Portfolio generating functions

Certain real-valued functions of the market weights $\mu_1(t), \ldots, \mu_n(t)$ can be used to construct dynamic portfolios that behave in a controlled manner. The *portfolio generating functions* that interest us most fall into two categories: smooth functions of the market weights, and smooth functions of the ranked market weights. Those portfolio generating functions that are smooth functions of the market weights can be used to create portfolios with returns that satisfy almost sure relationships relative to the market portfolio, and, hence, can be applied to situations in which arbitrage might be possible. Those functions that are smooth functions of the ranked market weights can be used to analyze the role of company *size* in portfolio behavior.

Suppose we are given a function $\mathbf{G} : U \to (0, \infty)$ which is defined and of class $\mathcal{C}^2$ on some open neighborhood U of Δ^n_+, and such that the mapping $x \mapsto x_i D_i \log \mathbf{G}(x)$ is bounded on U for all $i = 1, \cdots, n$. Consider also the portfolio $\pi(\cdot)$ with weights

$$\pi_i(t) = \Big(D_i \log \mathbf{G}(\mu(t)) + 1 - \sum_{j=1}^{n} \mu_j(t) D_j \log \mathbf{G}(\mu(t)) \Big) \cdot \mu_i(t)\,, \qquad 1 \le i \le n. \tag{11.1}$$

We call this the portfolio *generated by* $\mathbf{G}(\cdot)$. It can be shown that the relative wealth process of this portfolio, with respect to the market, is given by the *master formula*

$$\log \left(\frac{V^{\pi}(T)}{V^{\mu}(T)} \right) \; = \; \log \left(\frac{\mathbf{G}(\mu(T))}{\mathbf{G}(\mu(0))} \right) + \int_0^T \mathfrak{g}(t)\, dt\,, \qquad 0 \le T < \infty \tag{11.2}$$

with *drift process* $\mathfrak{g}(\cdot)$ given by

$$\mathfrak{g}(t) \; := \; \frac{-1}{2\mathbf{G}(\mu(t))} \sum_{i=1}^{n} \sum_{j=1}^{n} D^2_{ij} \mathbf{G}(\mu(t)) \mu_i(t) \mu_j(t) \tau^{\mu}_{ij}(t)\,. \tag{11.3}$$

The portfolio weights of (11.1) depend only on the market weights $\mu_1(t), \cdots, \mu_n(t)$, not on the covariance structure of the market. Therefore (11.1) can be implemented, and its associated wealth process $V^{\pi}(\cdot)$ observed through time, only in terms of the evolution of these market weights over $[0, T]$.

The covariance structure enters only in the computation of the drift term in (11.3). But the remarkable thing is that, in order to compute the cumulative effect $\int_0^T \mathfrak{g}(t)\, dt$ of this drift, there is no need to know or estimate this covariance structure at all, since (11.2) does this for us in the form $\int_0^T \mathfrak{g}(t)\, dt = \log \big(V^{\pi}(T) \mathbf{G}(\mu(0)) / V^{\mu}(T) \mathbf{G}(\mu(T)) \big)$, and in terms of quantities that are observable.

The proof of the very important "master formula" (11.2) is given in subsection 11.3 below. This can be skipped on first reading.

11.1 Remark. Suppose the function $\mathbf{G}(\cdot)$ is *concave*, or, more precisely, its Hessian $D^2\mathbf{G}(x) = \big(D^2_{ij} \mathbf{G}(x) \big)_{1 \le i,j \le n}$ has at most one positive eigenvalue for each $x \in U$ and, if a positive eigenvalue exists, the corresponding eigenvector is orthogonal to Δ^n_+. Then the portfolio $\pi(\cdot)$ generated by $\mathbf{G}(\cdot)$ as in (11.1) is long-only (i.e., each weight $\pi_i(\cdot)$ is non-negative), and the drift term $\mathfrak{g}(\cdot)$ is non-negative; if $\text{rank}\big(D^2\mathbf{G}(x)\big) > 1$ holds for each $x \in U$, then $\mathfrak{g}(\cdot)$ is positive.

For instance, the choice

1. $\mathbf{G}(\cdot) \equiv w$, a positive constant, generates the market portfolio; the choice
2. $\mathbf{G}(x) = w_1 x_1 + \cdots + w_n x_n$ generates the portfolio that buys at time $t = 0$, and holds up until time $t = T$, a fixed number of shares w_i in each stock $i = 1, \cdots, n$ (the market portfolio corresponds to the special case $w_1 = \cdots = w_n = w$); the choice
3. $\mathbf{G}(x) = \big(x_1 \cdots x_n \big)^{1/n}$, generates the *equally-weighted* portfolio $\pi_i(\cdot) \equiv 1/n$, $i = 1, \cdots, n$ with $\mathfrak{g}(\cdot) \equiv \gamma^*_{\pi}(\cdot)$; and the choice

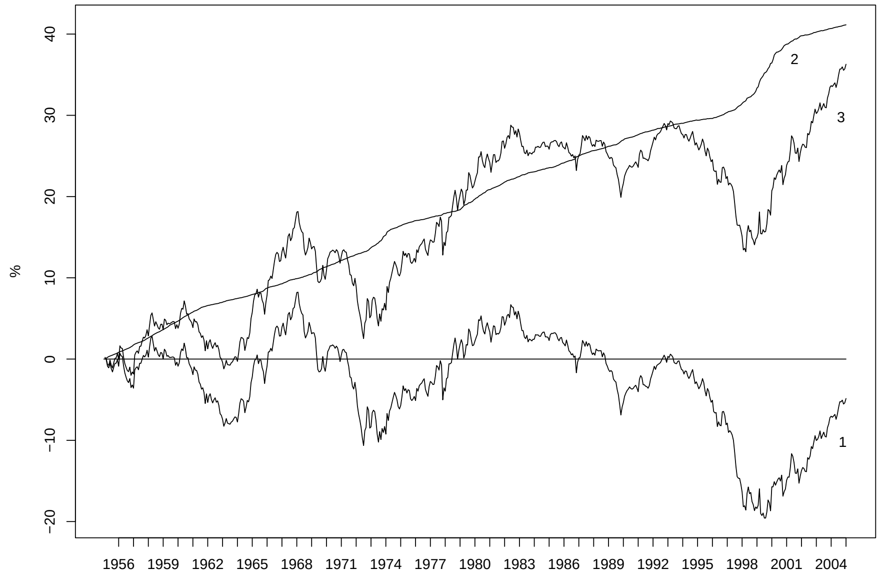

Figure 2: Simulation of a $\mathbf{G}_p$-weighted portfolio, 1956–2005
1: generating function; 2: drift process; 3: relative return.

4. $\mathbf{G}(x) = \big(x_1^p + \cdots + x_n^p\big)^{1/p}$, for some $0 < p < 1$, generates the *diversity-weighted* portfolio $\mu^{(p)}(\cdot)$, of (7.1) with $\mathfrak{g}(\cdot) \equiv (1-p)\gamma^*_{\mu^{(p)}}(\cdot)$.

5. Consider now the *Shannon entropy function* $\mathbf{H}(x) := 1 - \sum_{i=1}^n x_i \log x_i,\ x \in \Delta^n_+$ and, for any given $c \in (0,\infty)$, its modification

$$\mathbf{H}_c(x) := c + \mathbf{H}(x), \quad \text{which satisfies:} \quad c < \mathbf{H}_c(x) \le c + \log n, \ \ x \in \Delta^n_+. \tag{11.4}$$

This new, modified entropy function generates an *entropy-weighted* portfolio $\pi^c(\cdot)$ with weights and drift-process given, respectively, as

$$\pi^c_i(t) = \frac{\mu_i(t)}{\mathbf{H}_c(\mu(t))}\big(c - \log \mu_i(t)\big), \ \ 1 \le i \le n \quad \text{and} \quad \mathfrak{g}^c(t) = \frac{\gamma^*_\mu(t)}{\mathbf{H}_c(\mu(t))}\,. \tag{11.5}$$

To get some idea about the behavior of one of these portfolios with actual stocks, we ran a simulation of a diversity-weighted portfolio using the stock database from the Center for Research in Securities Prices (CRSP) at the University of Chicago. The data included 50 years of monthly values from 1956 to 2005 for exchange-traded stocks after the removal of closed-end funds, REITs, and ADRs not included in the S&P 500 Index. From this universe, we considered a cap-weighted large-stock index consisting of the largest 1000 stocks in the database. Against this index, we simulated the performance of the corresponding diversity-weighted portfolio, generated by $\mathbf{G}_p$ of Example 4 above with $p = 1/2$. No trading costs were included.

The results of the simulation are presented in Figure 2: Curve 1 is the change in the generating function, Curve 2 is the drift process, and Curve 3 is the relative return. Each curve shows the cumulative value of the monthly changes induced in the corresponding process by capital gains or losses in the stocks, so the curves are unaffected by monthly changes in the composition of the database. As can be seen, Curve 3 is the sum of Curves 1 and 2. The drift process $\int_0^{\cdot} \mathfrak{g}(t)\,dt$ was the dominant term over the period with a total contribution of about 40 percentage points to the relative return. The drift process $\mathfrak{g}(\cdot)$ was quite stable over the 50-year period, with the possible exception of the period around 2000, when "irrational exuberance" increased the volatility of the stocks as well as the intrinsic volatility of the entire market and, hence, increased the value of $\mathfrak{g}(\cdot) \equiv (1-p)\gamma^*_{\mu^{(p)}}(\cdot)$. The cumulative drift process $\int_0^\cdot \mathfrak{g}(t)\,dt$ here has been adjusted to account for "leakage"; see Remark 11.11 below.

11.1 Sufficient Intrinsic Volatility leads to Arbitrage

Broadly accepted practitioner wisdom upholds that sufficient volatility creates opportunities in a market. We shall try to put this intuition on a precise quantitative basis in Example 11.2 below, by identifying the excess growth rate of the market portfolio – which also measures the market's "intrinsic volatility", according to (3.8) and the discussion following it – as a quantity whose "availability" or "sufficiency" (boundedness away from zero) can lead to arbitrage opportunities relative to the market.

11.2 Example. Suppose now that in the market $\mathcal{M}$ there exists a constant $\zeta > 0$ such that

$$\frac{1}{T}\int_0^T \gamma^*_\mu(t)\,dt \geq \zeta \tag{11.6}$$

holds almost surely. For instance, this is the case when the excess growth rate of the market portfolio is bounded away from zero: that is, when

$$\gamma^*_\mu(t) \geq \zeta, \quad \forall\ \ 0 \leq t \leq T \tag{11.7}$$

holds almost surely, for some constant $\zeta > 0$.

Consider again the entropy-weighted portfolio $\pi^c(\cdot)$ of (11.5), namely

$$\pi^c_i(t) \;=\; \frac{\mu_i(t)\big(c - \log \mu_i(t)\big)}{\sum_{j=1}^n \mu_j(t)\big(c - \log \mu_j(t)\big)}\,, \qquad i = 1, \cdots, n\,, \tag{11.8}$$

now written in a form that makes plain its over-weighting of the small capitalization stocks, relative to the market portfolio. From (11.2), (11.5) and the inequalities of (11.4), one sees that the portfolio $\pi^c(\cdot)$ in (11.5) satisfies

$$\begin{aligned}\log\left(\frac{V^{\pi^c}(T)}{V^\mu(T)}\right) \;&=\; \log\left(\frac{\mathbf{H}_c(\mu(T))}{\mathbf{H}_c(\mu(0))}\right) + \int_0^T \frac{\gamma^*_\mu(t)}{\mathbf{H}_c(\mu(t))}\,dt \\ &> \;-\log\left(\frac{c + \mathbf{H}\big(\mu(0)\big)}{c}\right) + \frac{\zeta T}{c + \log n}\end{aligned} \tag{11.9}$$

almost surely. Thus, for every time horizon

$$T > \mathcal{T}_*(c) := \frac{1}{\zeta}\,\big(c + \log n\big)\log\left(\frac{c + \mathbf{H}\big(\mu(0)\big)}{c}\right),$$

or for that matter any

$$T > \mathcal{T}_* = \frac{1}{\zeta}\,\mathbf{H}\big(\mu(0)\big) \tag{11.10}$$

(since $\lim_{c\to\infty} \mathcal{T}_*(c) = \mathcal{T}_*$), and for $c > 0$ sufficiently large, the portfolio $\pi^c(\cdot)$ of (11.5) satisfies the conditions of (6.1) relative to the market $\rho(\cdot) \equiv \mu(\cdot)$, on the given time horizon $[0, T]$. It is straightforward that (6.3) is also satisfied, with $q = c/(c + \mathbf{H}(\mu(0))$. □

In particular, we have $\mathcal{L}^{\pi^c,\mu} \geq \zeta/(c + \log n) > 0$ a.s., under the condition (11.7) and with the notation of (6.2). Note also that *we have not assumed in this discussion any condition on the volatility structure of the market* (such as (1.14), or (3.10)), beyond the minimal assumption of (1.2).

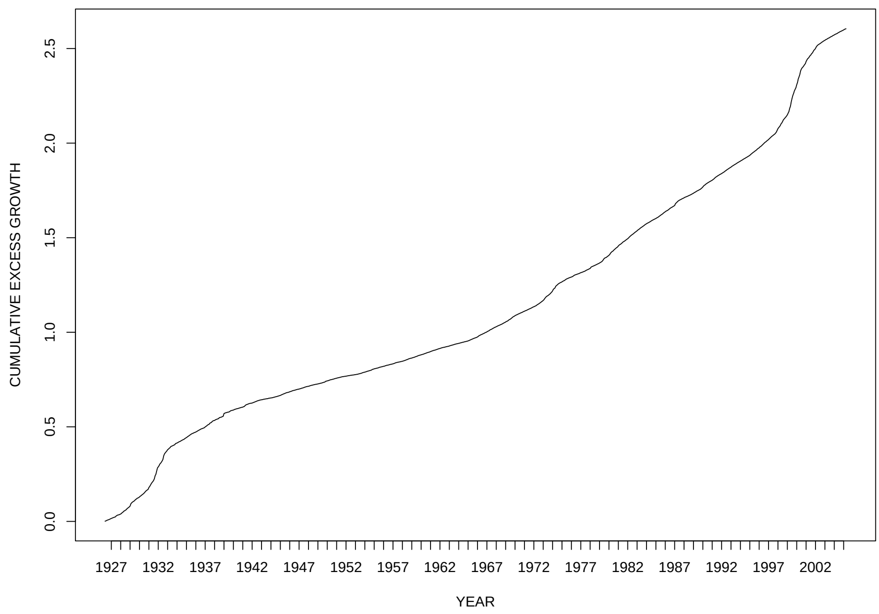

Figure 3: Cumulative excess growth $\int_0^{\cdot} \gamma_\mu^*(t)dt$.
U.S. market, 1927–2005

Figure 3 shows the cumulative market excess growth for the U.S. equities over most of the twentieth century. Note the conspicuous bumps in the curve, first in the Great Depression period in the early 1930s, then again in the "irrational exuberance" period at the end of the century. The data used for this chart come from the monthly stock database of the Center for Research in Securities Prices (CRSP) at the University of Chicago. The market we construct consists of the stocks traded on the New York Stock Exchange (NYSE), the American Stock Exchange (AMEX) and the NASDAQ Stock Market, after the removal of all REITs, all closed-end funds, and those ADRs not included in the S&P 500 Index. Until 1962, the CRSP data included only NYSE stocks. The AMEX stocks were included after July 1962, and the NASDAQ stocks were included at the beginning of 1973. The number of stocks in this market varies from a few hundred in 1927 to about 7500 in 2005.

This computation for Figure 3 does not need any estimation of covariance structure: from (11.9) we can express this cumulative excess growth

$$\int_0^{\cdot} \gamma_\mu^*(t)\, dt = \int_0^{\cdot} \mathbf{H}_c\big(\mu(t)\big)\, d\log\left(\frac{V^{\pi^c}(t)\, \mathbf{H}_c(\mu(0))}{V^{\mu}(t)\, \mathbf{H}_c(\mu(t))}\right) ,$$

just in terms of quantities that are observable in the market. The plot suggests that the U.S. market has exhibited a strictly increasing cumulative excess growth over this period.

11.3 Remark. Let us recall here our discussion of the conditions in (5.3): if the covariance matrix $a(\cdot)$ has all its eigenvalues bounded away from both zero and infinity, then the condition (11.7) (respectively, (11.6)) is equivalent to diversity (respectively, weak diversity) on $[0,T]$. The point of these conditions is that they guarantee the existence of relative arbitrage even when volatilities are unbounded and diversity fails. In the next section we shall study a concrete example of such a situation.

11.4 Remark. Major Open Question: Is the condition (11.7) sufficient for the existence of relative arbitrage over *arbitrary* (as opposed to sufficiently long) time horizons?

11.5 Remark. Major Open Questions: For $0 < p \leq 1$, introduce the quantity

$$\gamma^*_{\pi,p}(t) := \frac{1}{2} \sum_{i=1}^{n} \big(\pi_i(t)\big)^p \, \tau^{\pi}_{ii}(t) \tag{11.11}$$

which generalizes the excess growth rate of a portfolio $\pi(\cdot)$, in the sense $\gamma^*_{\pi,1}(\cdot) \equiv \gamma^*_{\pi}(\cdot)$. With $0 < p < 1$, consider the a.s. requirement

$$\Gamma(T) \leq \int_0^T \gamma^*_{p,\mu}(t)\, dt < \infty, \qquad \forall \;\; 0 \leq T < \infty, \tag{11.12}$$

where $\Gamma : [0,\infty) \to [0,\infty)$ is a continuous, strictly increasing function with $\Gamma(0) = 0$, $\Gamma(\infty) = \infty$. As shown in Proposition 3.8 of [FK] (2005), the condition (11.12) guarantees that the portfolio

$$\pi_i(t) := \frac{p\,\big(\mu_i(t)\big)^p}{\sum_{j=1}^n \big(\mu_j(t)\big)^p} + (1-p)\,\mu_i(t), \qquad i = 1, \cdots, n \tag{11.13}$$

is an arbitrage opportunity relative to the market, namely, that $\mathbb{P}\,[\, V^{\pi}(T) > V^{\mu}(T)\,] = 1$ holds over sufficiently long time-horizons: $T > \Gamma^{-1}\big((1/p) n^{1-p} \log n\big)$.

Some questions suggest themselves:

- Does (11.12) guarantee the existence of relative arbitrage opportunities over arbitrary time-horizons?
- Is there a result on the existence of relative arbitrage, that generalizes both Example 11.2 and the result outlined in (11.12), (11.13) of this Remark 11.5?
- What quantity, or quantities, might then be involved, in place of the market excess growth or its generalization (11.12)? Is there a "best" result of this type?

11.6 Remark. Open Question: We have presented a few portfolios that lead to arbitrage relative to the market; they are all functionally generated (F-G). Is there a "best" such example within that class? Are there similar examples of portfolios that are *not* functionally generated, nor trivial modifications thereof? How representative (or "dense") in this context is the class of F-G portfolios?

11.7 Remark. Open Question: Generalize the theory of F-G portfolios to the case of a market with a countable infinity ($n = \infty$) of assets, or to some other model with a variable, unbounded number of assets.

11.8 Remark. Open Question: What, if any, is the connection of F-G portfolios with the "universal portfolios" of Cover (1991) and Jamshidian (1992) ?

11.2 Rank, Leakage, and the Size Effect

An important generalization of the ideas and methods in this section concerns generating functions that record market weights not according to their name (or index) i, but according to their rank. To present this generalization, let us start by recalling the order statistics notation of (1.16), and consider for each $0 \leq t < \infty$ the random permutation $\big(p_t(1), \cdots, p_t(n)\big)$ of $(1, \cdots, n)$ with

$$\mu_{p_t(k)}(t) = \mu_{(k)}(t), \qquad \text{and} \quad p_t(k) < p_t(k+1) \quad \text{if} \quad \mu_{(k)}(t) = \mu_{(k+1)}(t) \tag{11.14}$$

for $k = 1, \ldots, n$. In words: $p_t(k)$ is the name (index) of the stock with the k^{th} largest relative capitalization at time t, and ties are resolved by resorting to the lowest index.

Using Itô's rule for convex functions of semimartingales (e.g. [KS] (1991), section 3.7), one can obtain the following analogue of (2.5) for the *ranked market-weights*

$$
\begin{aligned}
\frac{d\mu_{(k)}(t)}{\mu_{(k)}(t)} = & \Big(\gamma_{p_t(k)}(t) - \gamma^{\mu}(t) + \frac{1}{2}\tau^{\mu}_{(kk)}(t)\Big)\, dt + \frac{1}{2}\Big(d\mathfrak{L}^{k,k+1}(t) - d\mathfrak{L}^{k-1,k}(t)\Big) \\
& + \sum_{\nu=1}^{d} \big(\sigma_{p_t(k)\nu}(t) - \sigma^{\mu}_{\nu}(t)\big)\, dW_{\nu}(t)
\end{aligned}
\tag{11.15}
$$

for each $k = 1, \ldots, n-1$. Here the quantity $\mathfrak{L}^{k,k+1}(t) \equiv \Lambda_{\Xi_k}(t)$ is the semimartingale local time at the origin, accumulated by the non-negative process

$$
\Xi_k(t) := \log\big(\mu_{(k)}/\mu_{(k+1)}\big)(t), \qquad 0 \le t < \infty \tag{11.16}
$$

by the calendar time t; it measures the cumulative effect of the changes that have occurred during the time-interval $[0,t]$ between ranks k and $k+1$. We are also setting $\mathfrak{L}^{0,1}(\cdot) \equiv 0$, $\mathfrak{L}^{m,m+1}(\cdot) \equiv 0$ and $\tau^{\mu}_{(k\ell)}(\cdot) := \tau^{\mu}_{p_t(k)p_t(\ell)}(\cdot)$.

A derivation of this result, under appropriate conditions that we choose not to broach here, can be found on pp. 76-79 of Fernholz (2002); see also Banner & Ghomrasni (2006) for generalizations. With this setup, we have then the following generalization of the master formula (11.2): consider a function $\mathbf{G} : U \to (0,\infty)$ exactly as assumed there, written in the form

$$
\mathbf{G}(x_1, \cdots, x_n) = \mathcal{G}\big(x_{(1)}, \cdots, x_{(n)}\big), \qquad \forall \quad x \in U
$$

for some $\mathcal{G} \in \mathcal{C}^2(U)$ and U an open neighborhood of Δ^n_+. Then with the shorthand $\mu_{(\cdot)}(t) := \big(\mu_{p_t(1)}(t), \cdots, \mu_{p_t(n)}(t)\big)'$ and the notation

$$
\begin{aligned}
\Gamma(T) := & -\int_0^T \frac{1}{2\mathcal{G}\big(\mu_{(\cdot)}(t)\big)} \sum_{k=1}^{n}\sum_{\ell=1}^{n} D^2_{k\ell}\mathcal{G}\big(\mu_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau^{\mu}_{(k\ell)}(t)\, dt \\
& + \frac{1}{2}\sum_{k=1}^{n-1} \big(\underline{\pi}_{p_t(k+1)}(t) - \underline{\pi}_{p_t(k)}(t)\big)\, d\mathfrak{L}^{k,k+1}(t)\,,
\end{aligned}
\tag{11.17}
$$

one can show that the performance of the portfolio $\underline{\pi}(\cdot)$ in

$$
\underline{\pi}_{p_t(k)}(t) = \left(D_k \log \mathcal{G}\big(\mu_{(\cdot)}(t)\big) + 1 - \sum_{\ell=1}^{n} \mu_{(\ell)}(t) D_\ell \log \mathcal{G}\big(\mu_{(\cdot)}(t)\big)\right)\mu_{(k)}(t)\,, \tag{11.18}
$$

$1 \le k \le n$, relative to the market, is given as

$$
\log\left(\frac{V^{\underline{\pi}}(T)}{V^{\mu}(T)}\right) = \log\left(\frac{\mathcal{G}\big(\mu_{(\cdot)}(T)\big)}{\mathcal{G}\big(\mu_{(\cdot)}(0)\big)}\right) + \Gamma(T)\,, \qquad 0 \le T < \infty\,. \tag{11.19}
$$

We say that $\underline{\pi}(\cdot)$ is then the portfolio generated by the function $\mathcal{G}(\cdot)$. The detailed proof can be found in Fernholz (2002), pp. 79-83.

For instance, $\mathcal{G}(x) = x_{(1)}$ generates the portfolio $\underline{\pi}_{p_t(k)}(\cdot) = \delta_{1k}$, $k = 1, \cdots, n$ that invests only in the largest stock. The relative performance

$$
\log\left(\frac{V^{\underline{\pi}}(T)}{V^{\mu}(T)}\right) = \log\left(\frac{\mu_{(1)}(T)}{\mu_{(1)}(0)}\right) - \frac{1}{2}\,\mathfrak{L}^{1,2}(T), \quad 0 \le T < \infty
$$

of this portfolio will suffer in the long run, if there are many changes in leadership: in order for the biggest stock to do well relative to the market, it must crush all competition!

11.9 Example. The Size Effect: This is the tendency of small stocks to have higher long-term returns relative to their larger brethren. The formula of (11.19) offers a simple, structural explanation of this observed phenomenon, as follows.

Fix an integer $m \in \{2, \cdots, n-1\}$ and consider the functions $\mathbf{G}_L(x) = x_{(1)} + \cdots + x_{(m)}$ and $\mathbf{G}_S(x) = x_{(m+1)} + \cdots + x_{(n)}$. These generate, respectively, a large-stock portfolio

$$\zeta_{p_t(k)}(t) = \frac{\mu_{(k)}(t)}{\mathbf{G}_L(\mu(t))}, \quad k = 1, \cdots, m \quad \text{and} \quad \zeta_{p_t(k)}(t) = 0, \quad k = m+1, \cdots, n \tag{11.20}$$

and a small-stock portfolio

$$\eta_{p_t(k)}(t) = \frac{\mu_{(k)}(t)}{\mathbf{G}_S(\mu(t))}, \quad k = m+1, \cdots, n \quad \text{and} \quad \eta_{p_t(k)}(t) = 0, \quad k = 1, \cdots, m. \tag{11.21}$$

According to (11.19), the performances of these portfolios, relative to the market, are given by

$$\log\left(\frac{V^\zeta(T)}{V^\mu(T)}\right) = \log\left(\frac{\mathbf{G}_L(\mu(T))}{\mathbf{G}_L(\mu(0))}\right) - \frac{1}{2}\int_0^T \zeta_{(m)}(t)\, d\mathfrak{L}^{m,m+1}(t), \tag{11.22}$$

$$\log\left(\frac{V^\eta(T)}{V^\mu(T)}\right) = \log\left(\frac{\mathbf{G}_S(\mu(T))}{\mathbf{G}_S(\mu(0))}\right) + \frac{1}{2}\int_0^T \eta_{(m)}(t)\, d\mathfrak{L}^{m,m+1}(t), \tag{11.23}$$

respectively. Therefore,

$$\log\left(\frac{V^\eta(T)}{V^\zeta(T)}\right) = \log\left(\frac{\mathbf{G}_S(\mu(T))\mathbf{G}_L(\mu(0))}{\mathbf{G}_L(\mu(T))\mathbf{G}_S(\mu(0))}\right) + \int_0^T \frac{\zeta_{(m)}(t) + \eta_{(m)}(t)}{2}\, d\mathfrak{L}^{m,m+1}(t). \tag{11.24}$$

If there is "stability" in the market, in the sense that the ratio of the relative capitalization of small to large stocks remains stable over time, then the first term on the right-hand side of (11.24) does not change much, whereas the second term keeps increasing and accounts for the better relative performance of the small stocks. Note that this argument does not invoke at all any assumption about the putative greater riskiness of the smaller stocks.

The paper Fernholz & Karatzas (2006) studies conditions under which such stability in relative capitalizations prevails, and contains further discussion related to the "liquidity premium" for equities.

11.10 Remark. Estimation of Local Times: Hard as this might be to have guessed from the outset, the local times $\mathfrak{L}^{k,k+1}(\cdot) \equiv \Lambda_{\Xi_k}(\cdot)$ appearing in (11.15), (11.17) can be estimated in practice quite accurately; indeed, (11.22) gives

$$\mathfrak{L}^{m,m+1}(\cdot) = \int_0^{\cdot} \frac{2}{\zeta_{(m)}(t)}\, d\log\left(\frac{\mathbf{G}_L(\mu(t))}{\mathbf{G}_L(\mu(0))}\frac{V^\mu(t)}{V^\zeta(t)}\right), \quad m = 1, \cdots, n-1, \tag{11.25}$$

and the quantity on the right-hand side is completely observable.

11.11 Remark. Leakage in a Diversity-Weighted Index of Large Stocks: With m and $\zeta(\cdot)$ as in Example 11.9 and fixed $r \in (0,1)$, consider the *diversity-weighted, large-stock portfolio*

$$\mu^\sharp_{p_t(k)}(t) = \frac{\left(\mu_{(k)}(t)\right)^r}{\sum_{\ell=1}^m \left(\mu_{(\ell)}(t)\right)^r}, \quad 1 \le k \le m \quad \text{and} \quad \mu^\sharp_{p_t(k)}(t) = 0, \quad m+1 \le k \le n \tag{11.26}$$

generated by $\mathfrak{G}_r(x) = \left(\sum_{\ell=1}^m \left(x_{(\ell)}\right)^r\right)^{1/r}$, by analogy with (7.5), (7.1).

It can be shown that the performance of the portfolio in (11.26) is given by

$$\log\left(\frac{V^{\mu^\sharp}(T)}{V^\mu(T)}\right) = \log\left(\frac{\mathfrak{G}_r(\mu(T))}{\mathfrak{G}_r(\mu(0))}\right) + (1-r)\int_0^T \gamma^*_{\mu^\sharp}(t)\, dt - \int_0^T \frac{\mu^\sharp_{(m)}(t)}{2}\, d\mathfrak{L}^{m,m+1}(t)$$

relative to the market, and by

$$\log\left(\frac{V^{\mu^\sharp}(T)}{V^{\zeta}(T)}\right) = \log\left(\frac{\mathfrak{G}_r\big(\zeta_{(1)}(T),\cdots,\zeta_{(m)}(T)\big)}{\mathfrak{G}_r\big(\zeta_{(1)}(0),\cdots,\zeta_{(m)}(0)\big)}\right) + (1-r)\int_0^T \gamma^*_{\mu^\sharp}(t)\,dt - \frac{1}{2}\int_0^T \big(\mu^\sharp_{(m)}(t)-\zeta_{(m)}(t)\big)\,d\mathfrak{L}^{m,m+1}(t) \tag{11.27}$$

relative to the large-stock portfolio $\zeta(\cdot)$ of (11.20).

Because $\mu^\sharp_{(m)}(\cdot) \geq \zeta_{(m)}(\cdot)$ from (7.8) and the remark following it, the the last term in (11.27) is monotonically increasing function of T. It measures the "leakage" that occurs, when a capitalization-weighted portfolio is contained inside a larger market, and stocks cross-over (leak) from the cap-weighted to the market portfolio. For details of these derivations, see Fernholz (2002), pp. 84-88.

11.3 Proof of the "Master Equation" (11.2)

To ease notation somewhat, let us set $g_i(t) := D_i \log \mathbf{G}(\mu(t))$ and $N(t) := 1-\sum_{j=1}^n \mu_j(t)g_j(t)$, so (11.1) reads: $\pi_i(t) = \big(g_i(t)+N(t)\big)\mu_i(t)$, for $i=1,\cdots n$. Then the terms on the right-hand side of (3.9) become

$$\sum_{i=1}^n (\pi_i(t)/\mu_i(t))\,d\mu_i(t) = \sum_{i=1}^n g_i(t)\,d\mu_i(t) + N(t)\,d\Big(\sum_{i=1}^n \mu_i(t)\Big) = \sum_{i=1}^n g_i(t)\,d\mu_i(t)$$

and

$$\begin{aligned}\sum_{i=1}^n\sum_{j=1}^n \pi_i(t)\pi_j(t)\tau^\mu_{ij}(t) &= \sum_{i=1}^n\sum_{j=1}^n \big(g_i(t)+N(t)\big)\big(g_j(t)+N(t)\big)\mu_i(t)\mu_j(t)\tau^\mu_{ij}(t)\\ &= \sum_{i=1}^n\sum_{j=1}^n g_i(t)g_j(t)\mu_i(t)\mu_j(t)\tau^\mu_{ij}(t),\end{aligned}$$

the latter thanks to (2.8) and Lemma 3.1. Thus, (3.9) gives

$$d\log\left(\frac{V^\pi(t)}{V^\mu(t)}\right) = \sum_{i=1}^n g_i(t)\,d\mu_i(t) - \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n g_i(t)g_j(t)\mu_i(t)\mu_j(t)\tau^\mu_{ij}(t)\,dt. \tag{11.28}$$

On the other hand, $D^2_{ij}\log\mathbf{G}(x) = \big(D^2_{ij}G(x)/G(x)\big) - D_i\log\mathbf{G}(x)D_j\log\mathbf{G}(x)$, so we get

$$\begin{aligned}d\log\mathbf{G}(\mu(t) &= \sum_{i=1}^n g_i(t)\,d\mu_i(t) + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n D^2_{ij}\log\mathbf{G}(\mu(t))\,d\langle\mu_i,\mu_j\rangle(t)\\ &= \sum_{i=1}^n g_i(t)\,d\mu_i(t) + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \Big(\frac{D^2_{ij}\mathbf{G}(\mu(t))}{\mathbf{G}(\mu(t))} - g_i(t)g_j(t)\Big)\mu_i(t)\mu_j(t)\tau^\mu_{ij}(t)\,dt\end{aligned}$$

by Itô's rule in conjunction with (2.9). Comparing this last expression with (11.28) and recalling (11.3), we deduce (11.2), namely $d\log\mathbf{G}(\mu(t) = d\log\left(V^\pi(t)/V^\mu(t)\right) - \mathfrak{g}(t)dt$.

Part IV
Abstract Markets

The basic market model in (1.1) is too general for us to be able to draw many interesting conclusions. Hence, we would like to consider a more restricted class of models that still capture certain aspects of real equity markets, but are more analytically tractable than the general model (1.1). *Abstract markets* are relatively simple stochastic market models that exhibit selected characteristics of real equity markets, so that an understanding of these models will provide some insight into the behavior of actual markets. In particular, there are two classes of abstract markets that we shall discuss here: *volatility-stabilized markets* introduced in Fernholz & Karatzas (2005), and *rank-based models* exemplified by *Atlas models* and their generalizations, which first appeared in Fernholz (2002), with further development in Banner, Fernholz & Karatzas (2005).

12 Volatility-Stabilized Markets

Volatility-stabilized markets are remarkable because in these models the market itself behaves in a rather sedate fashion, viz., (exponential) Brownian motion with drift, while the individual stocks are going all over the place (in a rigorously defined manner, of course). These markets reflect the fact that in real markets, the smaller stocks tend to have greater volatility than the larger stocks.

Let us consider the abstract market model $\mathcal{M}$ with

$$d\log X_i(t) = \frac{\alpha}{2\mu_i(t)}\,dt + \frac{1}{\sqrt{\mu_i(t)}}\,dW_i(t), \qquad i = 1, \cdots, n, \tag{12.1}$$

where $\alpha \geq 0$ is a given real constant. The theory developed by Bass & Perkins (2002) shows that the resulting system of stochastic differential equations, for $i = 1, \ldots, n$,

$$dX_i(t) = \frac{1+\alpha}{2}\big(X_1(t) + \ldots + X_n(t)\big)\,dt + \sqrt{X_i(t)\big(X_1(t) + \ldots + X_n(t)\big)}\,dW_i(t), \tag{12.2}$$

determines the distribution of the Δ^n_+−valued diffusion process $\mathfrak{X}(\cdot) = \big(X_1(\cdot), \cdots, X_n(\cdot)\big)'$ uniquely; and that the conditions of (1.2), (6.4) are satisfied by the processes

$$b_i(\cdot) = (1+\alpha)/2\mu_i(\cdot)\,, \quad \sigma_{i\nu}(t) = (\mu_i(t))^{-1/2}\delta_{i\nu}\,, \quad r(\cdot) \equiv 0 \quad \text{and} \quad \theta_\nu(\cdot) = (1+\alpha)/2\sqrt{\mu_\nu(\cdot)}$$

for $1 \leq i, \nu \leq n$. The reader might wish to remark that condition (3.10) is satisfied in this case, in fact with $\varepsilon = 1$; but (1.14) fails.

The model of (12.1) assigns to all stocks log-drifts $\gamma_i(t) = \alpha/2\mu_i(t)$, and volatilities $\sigma_{i\nu}(t) = (\mu_i(t))^{-1/2}\delta_{i\nu}$ that are largest for the smallest stocks, and smallest for the largest stocks. Not surprisingly then, individual stocks fluctuate rather widely in a market of this type; in particular, *diversity fails* on every $[0, T]$; see Remarks 12.2 and 12.3.

Yet despite these fluctuations, the overall market has quite stable behavior. We call this phenomenon *stabilization by volatility* in the case $\alpha = 0$; and *stabilization by both volatility and drift* in the case $\alpha > 0$.

Indeed, the quantities $a_{\mu\mu}(\cdot)$, $\gamma^*_\mu(\cdot)$, $\gamma_\mu(\cdot)$ are computed from (2.7), (1.12), (1.11) as

$$a_{\mu\mu}(\cdot) \equiv 1, \quad \gamma^*_\mu(\cdot) \equiv \gamma^* := \frac{n-1}{2} > 0, \quad \gamma_\mu(\cdot) \equiv \gamma := \frac{(1+\alpha)n - 1}{2} > 0. \tag{12.3}$$

This, in conjunction with (2.2), computes the total market capitalization

$$X(t) \;=\; X_1(t) + \ldots + X_n(t) \;=\; X(0)\,e^{\gamma t + \mathcal{W}(t)}\,, \qquad 0 \leq t < \infty \tag{12.4}$$

as the exponential of the standard, one-dimensional Brownian motion $\mathcal{W}(\cdot) := \sum_{\nu=1}^n \int_0^\cdot \sqrt{\mu_\nu(s)}\,dW_\nu(s)$, plus drift $\gamma t > 0$. In particular, the overall market and the largest stock $X_{(1)}(\cdot) = \max_{1\leq i\leq n} X_i(\cdot)$ grow at the same, constant rate:

$$\lim_{T\to\infty} \frac{1}{T}\log X(T) = \lim_{T\to\infty} \frac{1}{T}\log X_{(1)}(T) = \gamma, \quad \text{a.s.} \tag{12.5}$$

On the other hand, according to Example 11.2 there exist in this model portfolios that lead to arbitrage opportunities relative to the market, at least on time horizons $[0,T]$ with $T \in (\mathcal{T}_*, \infty)$, where

$$\mathcal{T}_* := \frac{2\mathbf{H}(\mu(0))}{n-1} \leq \frac{2\log n}{n-1}.$$

To wit: *relative arbitrage can exist in non-diverse markets with unbounded volatilities.* The last upper bound in the above expression becomes small as the number of stocks in the market increases. Banner & Fernholz (2006) proved recently that arbitrage exists over arbitrary time-horizons in the market (12.1).

12.1 Bessel Processes

The crucial observation now, is that the solution of the system (12.1) can be expressed in terms of the squares of independent Bessel processes $\mathfrak{R}_1(\cdot), \ldots, \mathfrak{R}_n(\cdot)$ in dimension $\kappa := 2(1+\alpha) \geq 2$, and of an appropriate time change:

$$X_i(t) = \mathfrak{R}_i^2\big(\Lambda(t)\big), \qquad 0 \leq t < \infty, \quad i = 1, \ldots, n, \tag{12.6}$$

where

$$\Lambda(t) := \frac{1}{4}\int_0^t X(u)\,du \;=\; \frac{X(0)}{4}\int_0^t e^{\gamma s + \mathcal{W}(s)}\,ds, \qquad 0 \leq t < \infty \tag{12.7}$$

and

$$\mathfrak{R}_i(u) = \sqrt{X_i(0)} + \frac{\kappa - 1}{2}\int_0^u \frac{d\xi}{\mathfrak{R}_i(\xi)} + \mathfrak{W}_i(u), \qquad 0 \leq u < \infty. \tag{12.8}$$

Here, the driving processes $\mathfrak{W}_i(\cdot) := \int_0^{\Lambda^{-1}(\cdot)} \sqrt{\Lambda'(t)}\,dW_i(t)$ are independent, standard one-dimensional Brownian motions (e.g. [KS] (1991), pp. 157-162). In a similar vein, we have the representation

$$X(t) = \mathfrak{R}^2\big(\Lambda(t)\big), \quad 0 \leq t < \infty$$

of the total market capitalization, in terms of the Bessel process

$$\mathfrak{R}(u) = \sqrt{X(0)} + \frac{n\kappa - 1}{2}\int_0^u \frac{d\xi}{\mathfrak{R}(\xi)} + \mathfrak{W}(u), \qquad 0 \leq u < \infty \tag{12.9}$$

in dimension $n\kappa$, and of yet another one-dimensional Brownian motion $\mathfrak{W}(\cdot)$.

This observation provides a wealth of structure, which can be used then to study the asymptotic properties of the model (12.1).

12.1 Remark. For the case $\alpha > 0$ ($\kappa > 2$), we have the ergodic property

$$\lim_{u\to\infty} \frac{1}{\log u}\int_0^u \frac{d\xi}{\mathfrak{R}_i^2(\xi)} = \frac{1}{\kappa - 2} = \frac{1}{2\alpha}\,, \qquad \text{a.s.}$$

(a consequence of the Birkhoff ergodic theorem and of the strong Markov property of the Bessel process), as well as the *Lamperti representation*

$$\mathfrak{R}_i(u) = \sqrt{x_i}\, e^{\alpha\theta + \mathfrak{B}_i(\theta)}\Big|_{\theta = \int_0^u \mathfrak{R}_i^{-2}(\xi)d\xi}\,, \qquad 0 \leq u < \infty$$

for the Bessel process $\mathfrak{R}_i(\cdot)$ in terms of the exponential of a standard Brownian motion $\mathfrak{B}_i(\cdot)$ with positive drift $\alpha > 0$. Then one can deduce the a.s. properties

$$\lim_{u\to\infty} \frac{\log \mathfrak{R}_i(u)}{\log u} = \frac{1}{2}, \qquad \lim_{t\to\infty} \frac{1}{t}\log X_i(t) = \gamma, \tag{12.10}$$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T a_{ii}(t)\,dt = \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{dt}{\mu_i(t)} = \frac{2\gamma}{\alpha} = n + \frac{n-1}{\alpha}, \tag{12.11}$$

for each $i = 1, \ldots, n$. In particular, all stocks grow at the same asymptotic rate $\gamma > 0$ of (12.3), as does the entire market, and the model of (12.1) is coherent in the sense of Remark 5.1.

12.2 Remark. In the case $\alpha = 0$ ($\kappa = 2$), it can be shown that

$$\lim_{u\to\infty} \frac{\log \mathfrak{R}_i(u)}{\log u} = \frac{1}{2} \qquad \text{holds in probability ,}$$

but that we have almost surely:

$$\limsup_{u\to\infty} \frac{\log \mathfrak{R}_i(u)}{\log u} = \frac{1}{2}, \qquad \liminf_{u\to\infty} \frac{\log \mathfrak{R}_i(u)}{\log u} = -\infty.$$

It follows from this and (12.5) that

$$\lim_{t\to\infty} \frac{1}{t}\log X_i(t) = \gamma \qquad \text{holds in probability,} \tag{12.12}$$

and also that

$$\limsup_{t\to\infty} \frac{1}{t}\log X_i(t) = \gamma, \qquad \liminf_{t\to\infty} \frac{1}{t}\log X_i(t) = -\infty \tag{12.13}$$

hold almost surely. To wit, individual stocks can "crash" in this case, despite the overall stability of the market; and coherence now fails. (*Note:* this can be obtained from the zero-one law of Spitzer (1958): For a decreasing function $h(\cdot)$ we have

$$\mathbb{P}\big(\mathfrak{R}_i(u) \geq u^{1/2}h(u) \quad \text{for all} \;\; u > 0 \;\; \text{sufficiently large}\big) = \; 1 \;\; \text{or} \;\; 0,$$

depending on whether the series $\sum_{k=1}^{\infty} \big(k|\log h(k)|\big)^{-1}$ converges or diverges; see [FK] (2005) for details.)

12.3 Remark. In the case $\alpha = 0$ ($\kappa = 2$), it can be shown that

$$\lim_{u\to\infty} \mathbb{P}\big(\mu_i\big(\Lambda^{-1}(u)\big) > 1 - \delta\big) = \delta^{n-1}$$

holds for every $i = 1, \ldots, n$ and $\delta \in (0,1)$; here $\Lambda^{-1}(\cdot) = 4\int_0^{\cdot} \mathfrak{R}^{-2}(\xi)\,d\xi$ is the inverse of the time change $\Lambda(\cdot)$ in (12.7), and $\mathfrak{R}(\cdot)$ is the Bessel process in (12.9). It follows that this model is not diverse on $[0,\infty)$.

12.4 Remark. In the case $\alpha = 0$ ($\kappa = 2$), the exponential (strict) local martingale of (6.5) can be computed as

$$Z(t) = \frac{\sqrt{x_1 \ldots x_n}}{\mathfrak{R}_1(u)\ldots\mathfrak{R}_n(u)} \exp\left\{\frac{1}{2}\int_0^u \sum_{i=1}^n \mathfrak{R}_i^{-2}(\xi)\,d\xi\right\}\Bigg|_{u=\Lambda(t)}.$$

12.5 Remark. In the context of the volatility-stabilized model of this section with $p = 1/2$, the diversity-weighted portfolio

$$\mu_i^{(p)}(t) \; = \; \frac{\sqrt{\mu_i(t)}}{\sum_{j=1}^n \sqrt{\mu_j(t)}} \, , \qquad i = 1, \cdots, n$$

of (7.1) represents an arbitrage relative to the market portfolio, namely

$$\mathbb{P}\,[\, V^{\pi^{(p)}}(T) > V^{\mu}(T)\,] = 1\,, \quad \text{at least on time-horizons } [0,T] \text{ with } T > \frac{8\,\log n}{n-1}\,.$$

Furthermore, this diversity-weighted portfolio outperforms very significantly the market over long time-horizons:

$$\mathcal{L}^{\mu^{(p)},\mu} := \liminf_{T\to\infty} \frac{1}{T} \log\left(\frac{V^{\mu^{(p)}}(T)}{V^{\mu}(T)}\right) = \liminf_{T\to\infty} \frac{1}{2T}\int_0^T \gamma^*_{\mu^{(p)}}(t)\,dt \geq \frac{n-1}{8}\,, \qquad \text{a.s.}$$

Do the indicated limits exist? Can they be computed in closed form?

12.6 Remark. For the equally-weighted portfolio $\eta_i(\cdot) \equiv 1/n\,,\ i = 1,\cdots,n$ in the volatility-stabilized model with $\alpha > 0$ one can verify, using (11.2) with $\mathbf{G}(x) = (x_1\cdots x_n)^{1/n}$ and (12.11), that the limit

$$\mathcal{L}^{\eta,\mu} := \lim_{T\to\infty} \frac{1}{T}\log\left(\frac{V^{\eta}(T)}{V^{\mu}(T)}\right) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \gamma^*_{\eta}(t)\,dt$$

of (6.2) exists a.s. and equals

$$\mathcal{L}^{\eta,\mu} = \frac{n-1}{2n}\left(1+\frac{n-1}{n\alpha}\right).$$

In other words: equal-weighting, with its bias towards the small-capitalization stocks, outperforms *considerably* the market over long time-horizons.

12.7 Remark. In the context of this section, the portfolio

$$\widehat{\pi}_i(t) := \frac{1+\alpha}{2} - \left(\frac{n}{2}(1+\alpha)-1\right)\mu_i(t) = \lambda\eta_i(t) + (1-\lambda)\mu_i(t)\,, \qquad i = 1,\cdots,n$$

with $\lambda = n(1+\alpha)/2 \geq 1$ (that is, long in the equally-weighted $\eta(\cdot)$ of Exercise 12.7, and short in the market), has the *numéraire property*

$$V^{\pi}(\cdot)/V^{\widehat{\pi}}(\cdot) \quad \text{is a supermartingale, for every portfolio } \pi(\cdot)\,.$$

Does the a.s. limit $\mathcal{L}^{\widehat{\pi},\mu} := \lim_{T\to\infty} \frac{1}{T}\log\left(V^{\widehat{\pi}}(T)/V^{\mu}(T)\right)$ exist? If so, can its value be computed in closed form?

12.8 Remark. Open Question: For the diversity-weighted portfolio $\pi_i^c(\cdot)$ of (11.8), compute in the context of the volatility-stabilized model the expression

$$\mathcal{L}^{\pi^c,\mu} := \liminf_{T\to\infty} \frac{1}{T}\log\left(\frac{V^{\pi^c}(T)}{V^{\mu}(T)}\right) = \liminf_{T\to\infty} \frac{\gamma^*}{T}\int_0^T \frac{dt}{c+\mathbf{H}(\mu(t))}$$

of (6.2), using (11.8) and (12.3). But note already from these expressions that

$$\mathcal{L}^{\pi^c,\mu} \geq \frac{n-1}{2(c+\log n)} > 0 \quad \text{a.s.}\,,$$

indicating again a significant outperforming of the market over long time-horizons. Do the indicated limits exist, as one would expect?

12.9 Remark. Open Questions: For fixed $t \in (0,\infty)$, determine the distributions of $\mu_i(t)$, $i = 1,\cdots,n$ and of the largest $\mu_{(1)}(t) := \max_{1\leq i\leq n}\mu_i(t)$ and smallest $\mu_{(n)}(t) := \min_{1\leq i\leq n}\mu_i(t)$ market weights.

What can be said about the behavior of the averages $\frac{1}{T}\int_0^T \mu_{(k)}(t)dt$, particularly for the largest $(k=1)$ and the smallest $(k=n)$ stocks?

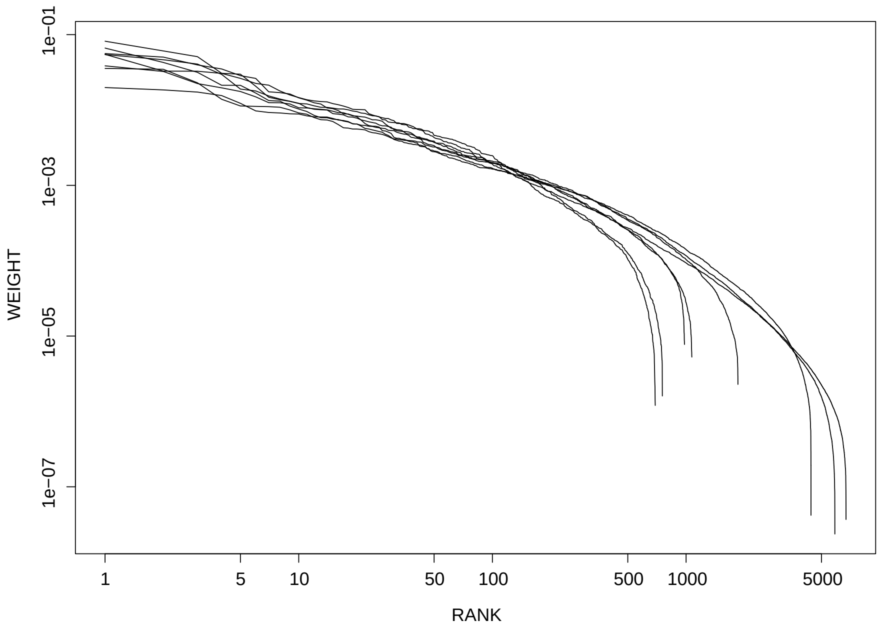

Figure 4: Capital distribution curves: 1929–1999.
The later the period, the longer the curve.

13 Ranked-Based Models

Size is one of the most important descriptive characteristics of financial assets. One can understand a lot about equity markets by observing, and trying to make sense of, the continual ebb and flow of small-, medium- and large-capitalization stocks in their midst. A particularly convenient way to study this feature is by looking at the evolution of the *capital distribution curve* $\log k \mapsto \log \mu_{(k)}(t)$; that is, the logarithms of the market weights arranged in descending order, versus the logarithms of their respective ranks (see also (13.13) below for the steady-state counterpart of this quantity). As shown in Figure 5.1 of Fernholz (2002), reproduced here as Figure 4, this log-log plot has exhibited remarkable stability over the decades of the last century.

It is of considerable importance, then, to have available models which describe this flow of capital and exhibit stability properties for capital distribution that are in at least broad agreement with these observations.

The simplest model of this type assigns growth rates and volatilities to the various stocks, not according to their names (the indices i) but according to their *ranks* within the market's capitalization. More precisely, let us pick real numbers $\gamma, g_1, \ldots, g_n$ and $\sigma_1 > 0, \ldots, \sigma_n > 0$, satisfying conditions that will be specified in a moment, and prescribe growth rates $\gamma_i(\cdot)$ and volatilities $\sigma_{i\nu}(\cdot)$

$$\gamma_i(t) \;=\; \gamma + \sum_{k=1}^{n} g_k 1_{\{X_i(t)=X_{p_t(k)}(t)\}} \qquad \sigma_{i\nu}(t) \;=\; \delta_{i\nu} \cdot \sum_{k=1}^{n} \sigma_k \, 1_{\{X_i(t)=X_{p_t(k)}(t)\}} \tag{13.1}$$

for $1 \le i, \nu \le n$ with $d = n$. We are using here the random permutation notation of (11.14), and we shall denote again by $\mathfrak{X}(\cdot) = \big(X_1(\cdot), \cdots, X_n(\cdot)\big)'$ the vector of stock-capitalizations.

It is clear intuitively that if such a model is to have some stability properties, it has to assign considerably higher growth rates to the smallest stocks than to the biggest ones. It turns out that the right conditions for stability are

$$g_1 < 0\,, \quad g_1 + g_2 < 0\,, \quad \ldots \quad , \quad g_1 + \cdots + g_{n-1} < 0\,, \quad g_1 + \cdots + g_n = 0\,. \tag{13.2}$$

These conditions are satisfied in the simplest model of this type, the *Atlas Model* that assigns

$$\gamma = g > 0\,, \qquad g_k = -g \;\text{ for }\; k = 1, \ldots, n-1 \qquad \text{and} \qquad g_n = (n-1)g\,, \tag{13.3}$$

thus $\gamma_i(t) = ng\, 1_{\{X_i(t) = X_{p_t(n)}(t)\}}$ in (13.1): zero growth rate goes to all the stocks but the smallest, which then becomes responsible for supporting the entire growth of the market.

Making these specifications amounts to postulating that the log-capitalizations $Y_i(\cdot) := \log X_i(\cdot)$ $i = 1, \cdots, n$ satisfy the system of stochastic differential equations

$$dY_i(t) \;=\; \Big(\gamma + \sum_{k=1}^n g_k 1_{\mathcal{Q}_i^{(k)}}(\mathfrak{Y}(t))\Big)dt \;+\; \sum_{k=1}^n \sigma_k 1_{\mathcal{Q}_i^{(k)}}(\mathfrak{Y}(t))\, dW_i(t)\,, \tag{13.4}$$

with $Y_i(0) = y_i = \log x_i$. Here $\big\{\mathcal{Q}_i^{(k)}\big\}_{1 \le i,k \le n}$ is a collection of polyhedral domains in $\mathbb{R}^n$, with the properties

$$\big\{\mathcal{Q}_i^{(k)}\big\}_{1\le i\le n} \quad \text{is a partition of } \mathbb{R}^n, \;\; \text{for each fixed } k\,,$$

$$\big\{\mathcal{Q}_i^{(k)}\big\}_{1\le k\le n} \quad \text{is a partition of } \mathbb{R}^n, \;\; \text{for each fixed } \; i\,,$$

and the interpretation

$$\mathrm{y} = (y_1, \ldots, y_n) \in \mathcal{Q}_i^{(k)} \quad \text{means that } \; y_i \; \text{ is ranked } \; k^{\text{th}} \text{ among } \; y_1, \ldots, y_n\,.$$

As long as the vector of log-capitalizations $\mathfrak{Y}(\cdot) = \big(Y_1(\cdot), \cdots, Y_n(\cdot)\big)'$ is in the polyhedron $\mathcal{Q}_i^{(k)}$, the equation (13.3) posits that the coördinate process $Y_i(\cdot)$ evolves like a Brownian motion with drift $\gamma + g_k$ and variance σ_k^2. (Ties are resolved by resorting to the lowest index i; for instance, $\mathcal{Q}_i^{(1)}$, $1 \le i \le n$ corresponds to the partition $\mathcal{Q}_i$ of $(0,\infty)^n$ of section 9, right below (9.3); and so on.)

The theory of Bass & Pardoux (1987) guarantees that this system has a weak solution, which is unique in distribution; once this solution has been constructed, we obtain stock capitalizations as $X_i(\cdot) = e^{Y_i(\cdot)}$ that satisfy (1.4) with the specifications of (13.1).

An immediate observation from (13.3) is that the sum $Y(\cdot) := \sum_{i=1}^n Y_i(\cdot)$ of log-capitalizations satisfies

$$Y(t) \;=\; y + n\gamma t + \sum_{k=1}^n \sigma_k B_k(t)\,, \qquad 0 \le t < \infty$$

with $y := \sum_{i=1}^n y_i$, and $B_k(\cdot) := \sum_{i=1}^n \int_0^\cdot 1_{\mathcal{Q}_i^{(k)}}(\mathfrak{Y}(s))dW_i(s)\,,\; k = 1, \ldots n$ independent scalar Brownian motions. Thus, the strong law of large numbers implies

$$\lim_{T\to\infty} \frac{1}{T} \sum_{i=1}^n Y_i(T) = n\gamma\,, \qquad \text{a.s.}$$

Then it takes a considerable amount of work (see Appendix in [BFK] 2005), in order to strengthen this result to

$$\lim_{T\to\infty} \frac{1}{T} \log X_i(T) = \lim_{T\to\infty} \frac{Y_i(T)}{T} = \gamma \quad \text{a.s., for every } \; i = 1, \ldots, n\,. \tag{13.5}$$

13.1 Remark. Using (13.5), it can be shown that the model specified by (1.5), (13.1) is coherent in the sense of Remark 5.1.

13.2 Remark. Taking Turns in the Various Ranks. From (13.4), (13.5) and the strong law of large numbers for Brownian motion, we deduce that the quantity $\sum_{k=1}^n g_k \left(\frac{1}{T}\int_0^T 1_{\mathcal{Q}_i^{(k)}}(\mathfrak{Y}(t))\, dt\right)$

converges a.s. to zero, as $T \to \infty$. For the Atlas model in (13.3), this expression becomes $g\left(\frac{n}{T}\int_0^T 1_{\mathcal{Q}_i^{(n)}}(\mathfrak{Y}(t))\,dt - 1\right)$, and we obtain

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T 1_{\mathcal{Q}_i^{(n)}}(\mathfrak{Y}(t))\,dt \;=\; \frac{1}{n} \quad \text{a.s., for every } \; i = 1,\ldots,n.$$

Namely, each stock spends roughly $(1/n)^{\text{th}}$ of the time, acting as "Atlas".

Again with considerable work, this is strengthened in [BFK] (2005) to the statement

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T 1_{\mathcal{Q}_i^{(k)}}\big(\mathfrak{Y}(t)\big)\,dt \;=\; \frac{1}{n}, \qquad \text{a.s., for every } \; 1 \le i, k \le n, \tag{13.6}$$

valid not just for the Atlas model, but under the more general conditions of (13.2). Thanks to the symmetry inherent in this model, each stock spends roughly $(1/n)^{\text{th}}$ of the time in any given rank; see Proposition 2.3 in [BFK] (2005).

13.1 Ranked Price Processes

For many purposes in the study of these models, it makes sense to look at the ranked log-capitalization processes

$$Z_k(t) := \sum_{i=1}^{n} Y_i(t)\cdot 1_{\mathcal{Q}_i^{(k)}}(\mathfrak{Y}(t)), \quad 0 \le t < \infty \tag{13.7}$$

for $1 \le k \le n$. From these, we get the ranked capitalizations via $X_{(k)}(t) = e^{Z_k(t)}$, with notation similar to (1.16). Using an extended Tanaka-type formula, as we did in (11.15), it can be seen that the processes of (13.7) satisfy

$$Z_k(t) \;=\; Z_k(0) + (g_k+\gamma)t + \sigma_k B_k(t) + \frac{1}{2}\Big(\mathfrak{L}^{k,k+1}(t) - \mathfrak{L}^{k-1,k}(t)\Big)\,, \quad 0 \le t < \infty \tag{13.8}$$

in that notation. Here, as in subsection 11.2, the continuous and increasing process $\mathfrak{L}^{k,k+1}(\cdot) := \Lambda_{\Xi_k}(\cdot)$ is the semimartingale local time at the origin of the continuous, non-negative process

$$\Xi_k(\cdot) \;=\; Z_k(\cdot) - Z_{k+1}(\cdot) \;=\; \log\big(\mu_{(k)}(\cdot)/\mu_{(k+1)}(\cdot)\big)$$

of (11.16) for $k = 1, \cdots, n-1$; and we make again the convention $\mathfrak{L}^{0,1}(\cdot) \equiv \mathfrak{L}^{n,n+1}(\cdot) \equiv 0$.

These local times play a big rôle in the analysis of this model. The quantity $\mathfrak{L}^{k,k+1}(T)$ represents again the cumulative amount of change between ranks k and $k+1$ that occurs over the time interval $[0,T]$. Of course, in a model such as the one studied here, the intensity of changes for the smaller stocks should be higher than for the larger stocks.

This is borne out by experiment: as we saw in Remark 11.10 it turns out, somewhat surprisingly, that these local times can be estimated based only on observations of relative market weights and of the performance of simple portfolios over $[0,T]$; and that they exhibit a remarkably linear increase, with positive rates that grow with k, as we see in Figure 5, reproduced from Fernholz (2002), Figure 5.2.

The analysis of the present model agrees with these observations: it follows from (13.5) and the dynamics of (13.8) that, for $k = 1,\ldots,n-1$, we have

$$\lim_{T\to\infty} \frac{1}{T}\,\mathfrak{L}^{k,k+1}(T) \;=\; \lambda_{k,k+1} \;:=\; -2\big(g_1+\ldots+g_k\big) > 0, \qquad \text{a.s.} \tag{13.9}$$

Our stability condition guarantees that these partial sums are positive – as indeed the limits on the right-hand side of (13.9) ought to be; and in typical examples, such as the Atlas model of (13.3) where $\lambda_{k,k+1} = kg$, they do increase with k, as suggested by Figure 5.

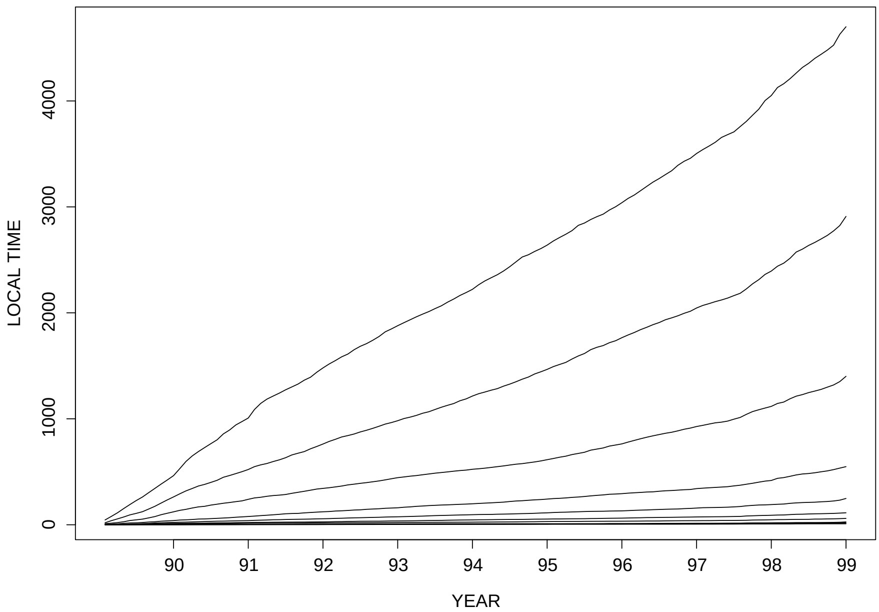

Figure 5: $\mathfrak{L}^{k,k+1}(\cdot)$, $k = 10, 20, 40, \cdots, 5120$.

13.2 Some Asymptotics

A slightly more careful analysis of these local times reveals that the non-negative semimartingale $\Xi_k(\cdot)$ of (11.16) can be cast in the form of a *Skorohod problem*

$$\Xi_k(t) = \Xi_k(0) + \Theta_k(t) + \Lambda_{\Xi_k}(t), \qquad 0 \le t < \infty,$$

as the reflection, at the origin, of the semimartingale

$$\Theta_k(t) = (g_k - g_{k+1})\, t - \frac{1}{2}\Big(\mathfrak{L}^{k-1,k}(t) + \mathfrak{L}^{k+1,k+2}(t)\Big) + \mathbf{s}_k \widetilde{W}^{(k)}(t),$$

where $\mathbf{s}_k := \big(\sigma_k^2 + \sigma_{k+1}^2\big)^{1/2}$ and $\widetilde{W}^{(k)}(\cdot) := \big(\sigma_k B_k(\cdot) - \sigma_{k+1} B_{k+1}(\cdot)\big)/\mathbf{s}_k$ is standard Brownian Motion.

As a result of these observations and of (13.9), we conclude that the process $\Xi_k(\cdot)$ behaves asymptotically like Brownian motion with drift $g_k - g_{k+1} - \frac{1}{2}\,(\lambda_{k-1,k} + \lambda_{k,k+1}) = -\lambda_{k,k+1} < 0$, variance $\mathbf{s}_k^2$, and reflection at the origin. Consequently,

$$\lim_{t\to\infty} \log\left(\frac{\mu_{(k)}(t)}{\mu_{(k+1)}(t)}\right) = \lim_{t\to\infty} \Xi_k(t) = \xi_k\,, \quad \text{in distribution} \tag{13.10}$$

where, for each $k = 1, \ldots, n-1$ the random variable ξ_k has an exponential distribution

$$\mathbb{P}(\xi_k > x) = e^{-r_k x},\ x \ge 0 \quad \text{with parameter} \quad r_k := \frac{2\lambda_{k,k+1}}{\mathbf{s}_k^2} = -\frac{4(g_1 + \cdots + g_k)}{\sigma_k^2 + \sigma_{k+1}^2} > 0. \tag{13.11}$$

13.3 The Steady-State Capital Distribution Curve

We also have from (13.10) the strong law of large numbers

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T g\big(\Xi_k(t)\big)\, dt = \mathbb{E}\big(g(\xi_k)\big), \qquad \text{a.s.}$$

for every rank k, and every measurable function $g : [0, \infty) \to \mathbb{R}$ which satisfies $\int_0^\infty |g(x)| e^{-r_k x} dx < \infty$; see Khas'minskii (1960). In particular,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \log\left(\frac{\mu_{(k)}(t)}{\mu_{(k+1)}(t)} \right) dt \;=\; \mathbb{E}\big(\xi_k\big) = \frac{1}{r_k} = \frac{\mathbf{s}_k^2}{2\lambda_{k,k+1}}, \quad \text{a.s.} \tag{13.12}$$

This observation provides a tool for studying the *steady-state capital distribution curve*

$$\log k \longmapsto \lim_{T\to\infty} \frac{1}{T} \int_0^T \log \mu_{(k)}(t)\, dt =: \mathfrak{m}(k), \quad k = 1, \cdots, n-1 \tag{13.13}$$

alluded to at the beginning of this section (more on the existence of this limit in the next subsection). To estimate the slope $\mathfrak{q}(k)$ of this curve at the point $\log k$, we use (13.12) and the estimate $\log(k+1) - \log k \approx 1/k$, to obtain in the notation of (13.11):

$$\mathfrak{q}(k) \;\approx\; \frac{\mathfrak{m}(k) - \mathfrak{m}(k+1)}{\log k - \log(k+1)} = -\frac{k}{r_k} = \frac{k\big(\sigma_k^2 + \sigma_{k+1}^2\big)}{4(g_1 + \cdots + g_k)} < 0\,. \tag{13.14}$$

Consider now an Atlas model as in (13.3). With equal variances $\sigma_k^2 = \sigma^2 > 0$, this slope is the constant $\mathfrak{q}(k) \approx -\sigma^2/2g$, and the steady-state capital distribution curve can be approximated by a straight *Pareto* line.

On the other hand, with variances of the form $\sigma_k^2 = \sigma^2 + ks^2$ for some $s^2 > 0$, growing linearly with rank, we get for large k the approximate slope

$$\mathfrak{q}(k) \approx -\frac{1}{2g}\big(\sigma^2 + ks^2\big), \qquad k = 1 . \cdots, n-1$$

Such linear growth is suggested by Figure 5.5 in Fernholz (2002), which is reproduced here as Figure 6. This would imply a *decreasing and concave* steady-state capital distribution curve, whose (negative) slope becomes more and more pronounced in magnitude with increasing rank, much in accord with the features of Figure 4.

13.3 Remark. Estimation of Parameters in this Model. Let us remark that (13.9) provides a method for obtaining estimates $\widehat{\lambda}_{k,k+1}$ of the parameters $\lambda_{k,k+1}$, from the observable random variables $\mathfrak{L}^{k,k+1}(T)$ that measure cumulative change between ranks k and $k+1$; recall Remark 11.10 once again. Then estimates of the parameters g_k follow, as $\widehat{g}_k = \big(\widehat{\lambda}_{k-1,k} - \widehat{\lambda}_{k,k+1}\big)/2$; and the parameters $\mathbf{s}_k^2 = \sigma_k^2 + \sigma_{k+1}^2$ can be estimated from (13.12) and from the increments of the observable capital distribution curve of (13.13), namely $\widehat{\mathbf{s}}_k^2 = 2\widehat{\lambda}_{k,k+1}\big(\mathfrak{m}(k) - \mathfrak{m}(k+1)\big)$. For the decade 1990-99, these estimates are presented in Figure 6.

Finally, we make the following selections for estimating the variances:

$$\widehat{\sigma}_k^2 = \frac{1}{4}\left(\widehat{\mathbf{s}}_{k-1}^2 + \widehat{\mathbf{s}}_k^2\right), \quad k = 2, \cdots, n-1, \qquad \widehat{\sigma}_1^2 = \frac{1}{2}\widehat{\mathbf{s}}_1^2, \quad \widehat{\sigma}_n^2 = \frac{1}{2}\widehat{\mathbf{s}}_{n-1}^2.$$

13.4 Stability of the Capital Distribution

Let us now go back to (13.10); it can be seen that this leads to the convergence of the ranked market weights

$$\lim_{t\to\infty} \big(\mu_{(1)}(t), \ldots, \mu_{(n)}(t)\big) = (M_1, \ldots, M_n), \quad \text{in distribution} \tag{13.15}$$

to the random variables

$$M_n := \big(1 + e^{\xi_{m-1}} + \cdots + e^{\xi_{n-1}+\cdots+\xi_1}\big)^{-1}, \quad \text{and} \quad M_k := M_n\, e^{\xi_{n-1}+\cdots+\xi_k} \tag{13.16}$$

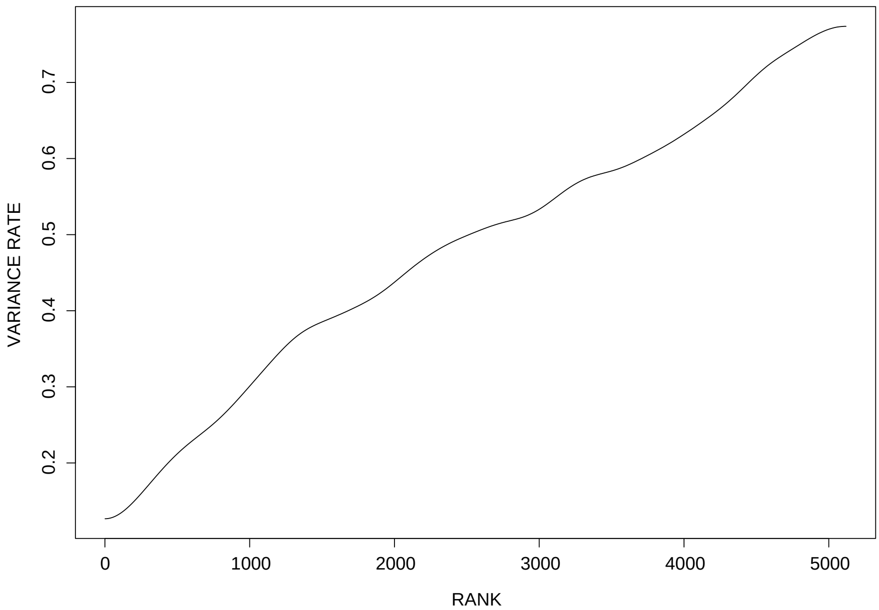

Figure 6: Smoothed annualized values of $\widehat{\mathbf{s}}_k^2$ for $k = 1, \ldots, 5119$.
Calculated from 1990–1999 data.

for $k = 1, \ldots, n-1$. These are the long-term (steady-state) relative weights of the various stocks in the market, ranked from largest, M_1, to smallest, M_n. Again, we have from (13.15) the strong law of large numbers

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f\big(\mu_{(1)}(t), \ldots, \mu_{(n)}(t)\big) dt \;=\; \mathbb{E}\big(f(M_1, \ldots, M_n)\big), \qquad \text{a.s.} \tag{13.17}$$

for every bounded and measurable $f : \Delta_+^n \to \mathbb{R}$. Note that (13.12) is a special case of this result, and that the function $\mathfrak{m}(\cdot)$ of (13.13) takes the form

$$\mathfrak{m}(k) \;=\; \mathbb{E}\big(\log(M_k)\big) \;=\; \sum_{\ell=k}^{n-1} \frac{1}{r_\ell} \;-\; \mathbb{E}\big(\log(1 + e^{\,\xi_{n-1}} + \cdots + e^{\,\xi_{n-1}+\cdots+\xi_1})\big). \tag{13.18}$$

This is the good news; the bad news is that we do not know the joint distribution of the exponential random variables $\xi_1, \cdots, \xi_{n-1}$ in (13.10), so we cannot find that of $M_1, \cdots, M_n$ either. In particular, we cannot pin down the steady-state capital distribution function of (13.18), though we *do* know precisely its increments $\mathfrak{m}(k+1) - \mathfrak{m}(k) = -(1/r_k)$ and thus are able to estimate the slope of the steady-state capital distribution curve, as indeed we did in (13.14). In [BFK] (2005) a simple, certainty-equivalent approximation of the steady-state ranked market weights of (13.16) is carried out, and is used to study in detail the behavior of simple portfolios in such a model.

13.4 Remark. Major Open Question: What can be said about the joint distribution of the long-term (steady-state) relative market weights of (13.16)? Can it be characterized, computed, or approximated in a good way? What can be said about the fluctuations of the random variables $\log(M_k)$ with respect to their means $\mathfrak{m}(k)$ in (13.18)?

13.5 Remark. Research Question and Conjecture: Study the steady-state capital distribution curve of the volatility-stabilized model in (12.1). With $\alpha > 0$, check the validity of the following conjecture: the slope

$$\mathfrak{q}(k) \approx (\mathfrak{m}(k) - \mathfrak{m}(k+1))/(\log k - \log(k+1))$$

of the capital distribution $\mathfrak{n}(\cdot)$ at $\log k$, should be given as

$$\mathfrak{q}(k) \approx -4\gamma k \mathfrak{h}_k \,, \qquad \mathfrak{h}_k := \mathbb{E}\left(\frac{\log Q_{(k)} - \log Q_{(k+1)}}{Q_{(1)} + \cdots + Q_{(n)}} \right),$$

where $Q_{(1)} \geq \cdots \geq Q_{(n)}$ are the order statistics of a random sample from the chi-square distribution with $\kappa = 2(1+\alpha)$ degrees of freedom.

If this conjecture is correct, does $k\mathfrak{h}_k$ increase with k?

14 Some Concluding Remarks

We have surveyed a framework, called *Stochastic Portfolio Theory*, for studying the behavior of portfolio rules and for modeling and analyzing equity market structure. We have also exhibited simple conditions, such as "diversity" and "availability of intrinsic volatility", which can lead to arbitrages relative to the market.

These conditions are *descriptive* in nature, and can be tested from the predictable characteristics of the model posited for the market. In contrast, familiar assumptions, such as the existence of an equivalent martingale measure (EMM), are *normative* in nature; they *cannot* be decided on the basis of predictable characteristics in the model. In this vein, see example in section 3.3 of [KK] (2006).

The existence of such relative arbitrage is not the end of the world. Under reasonably general conditions, one can still work with appropriate "deflators" for the purposes of hedging contingent claims and of portfolio optimization, as we have tried to illustrate in section 10.

Considerable computational tractability is lost, as the marvelous tool that is the EMM goes out the window. Nevertheless, big swaths of the field of Mathematical Finance remain totally or mostly intact; and completely new areas and issues, such as those of the "Abstract Markets" in Part IV of this survey, thrust themselves onto the scene.

15 Acknowledgements

We are indebted to Professor Alain Bensoussan for suggesting to us that we write this survey paper.

The paper is an expanded version of the Lukacs Lectures, given by one of us at Bowling Green University in May-June 2006. We are indebted to our hosts at Bowling Green, Ohio for the invitation to deliver the lectures, for their hospitality, their interest, and their incisive comments during the lectures; these helped us sharpen both our understanding, and improve the exposition of the paper.

Many thanks are due to Constantinos Kardaras for going over an early version of the manuscript and offering many valuable suggestions; to Adrian Banner for his comments on a later version; and to Mihai Sîrbu for helping us simplify *and* sharpen some of our results, and for catching several typos in the near-final version of the paper.

16 References

Banner, A., Fernholz, D. (2006) Short-term arbitrage in volatility-stabilized markets. *Preprint.*

Banner, A., Fernholz, R. & Karatzas, I. [BFK] (2005) On Atlas models of equity markets. *Annals of Applied Probability* **15**, 2296-2330.

Banner, A. & Ghomrasni, R. (2006) Local times of ranked continuous semimartingales. *Preprint.*

Bass, R. & Pardoux, E. (1987) Uniqueness of diffusions with piecewise constant coëfficients. *Probability Theory & Related Fields* **76**, 557-572.

Bass, R. & Perkins, E. (2002) Degenerate stochastic differential equations with Hölder-continuous coëfficients and super-Markov chains. *Transactions of the American Mathematical Society* **355**, 373-405.

Cover, T. (1991) Universal portfolios. *Mathematical Finance* **1**, 1-29.

Duffie, D. (1992) *Dynamic Asset Pricing Theory.* Princeton University Press, Princeton.

Fernholz, E.R. (1999) On the diversity of equity markets. *Journal of Mathematical Economics* **31**, 393-417.

Fernholz, E.R. (1999a) Portfolio generating functions. In M. Avellaneda (ed.), *Quantitative Analysis in Financial Markets*, River Edge, NJ. World Scientific.

Fernholz, E.R. (2001) Equity portfolios generated by functions of ranked market weights. *Finance & Stochastics* **5**, 469-486.

Fernholz, E.R. (2002) *Stochastic Portfolio Theory.* Springer-Verlag, New York.

Fernholz, E.R. & Karatzas, I. [FK] (2005) Relative arbitrage in volatility-stabilized markets. *Annals of Finance* **1**, 149-177.

Fernholz, E.R. & Karatzas, I. (2006) The implied liquidity premium for equities. *Annals of Finance* **2**, 87-99.

Fernholz, E.R., Karatzas, I. & Kardaras, C. [FKK] (2005). Diversity and arbitrage in equity markets. *Finance & Stochastics* **9**, 1-27.

Heath, D., Orey, S., Pestien, V. & Sudderth, W.D. (1987) Maximizing or minimizing the expected time to reach zero. *SIAM Journal on Control & Optimization* **25**, 195-205.

Jamshidian, F. (1992) Asymptotically optimal portfolios. *Mathematical Finance* **3**, 131-150.

Karatzas, I. & Kardaras, C. [KK] (2006) The numéraire portfolio and arbitrage in semimartingale markets. *Preprint.*

Karatzas, I., Lehoczky, J.P., Shreve, S.E. & Xu, G.L. (1991) Martingale and duality methods for utility maximization in an incomplete market. *SIAM Journal on Control & Optimization* **29**, 702-730.

Karatzas, I. & Shreve, S.E. [KS] (1991) *Brownian Motion and Stochastic Calculus.* Second Edition. Springer-Verlag, New York.

Karatzas, I. & Shreve, S.E. [KS] (1998) *Methods of Mathematical Finance.* Springer-Verlag, New York.

Kardaras, C. (2003) Stochastic Portfolio Theory in Semimartingale Markets. *Unpublished Manuscript, Columbia University.*

Kardaras, C. (2006) Personal communication.

Khas'minskii, R.Z. (1960) Ergodic properties of recurrent diffusion processes, and stabilization of the solution to the Cauchy problem for parabolic equations. *Theory of Probability and Its Applications* **5**, 179-196.

Lowenstein, M. & Willard, G.A. (2000.a) Local martingales, arbitrage and viability. *Economic Theory* **16**, 135-161.

Lowenstein, M. & Willard, G.A. (2000.b) Rational equilibrium asset-pricing bubbles in continuous trading models. *Journal of Economic Theory* **91**, 17-58.

Markowitz, H. (1952) Portfolio selection. *Journal of Finance* **7**, 77-91.

Osterrieder, J. & Rheinländer, Th. (2006) A note on arbitrage in diverse markets. *Annals of Finance* **2**, 287-301.

Pestien, V. & Sudderth, W.D. (1985) Continuous-time red-and-black: how to control a diffusion to a goal. *Mathematics of Operations Research* **10**, 599-611.

Platen, E. (2002) Arbitrage in continuous complete markets. *Advances in Applied Probability* **34**, 540-558.

Platen, E. (2006) A benchmark approach to Finance. *Mathematical Finance* **16**, 131-151.

Spitzer, F. (1958) Some theorems concerning two-dimensional Brownian motion. *Transactions of the American Mathematical Society* **87**, 187-197.

Sudderth, W.D. & Weerasinghe, A. (1989) Controlling a process to a goal in finite time. *Mathematics of Operations Research* **14**, 400-409.