On *Stochastic Portfolio Theory* by E. R. Fernholz

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The research monograph *Stochastic Portfolio Theory* by E. R. Fernholz presents a novel mathematical methodology for analyzing portfolio behavior and stock market structure. Stochastic portfolio theory is descriptive rather than normative, and hence it differs from the current doctrine of mathematical and quantitative finance. Descriptive theories offer explanations for observed phenomena and predictions for the outcomes of future experiments; the theories of natural science are descriptive theories. The central theories of finance are normative theories: general equilibrium, dynamic asset pricing, no-arbitrage. These normative theories are based on assumed ideal behavior regarding the interaction of the participants and agents in the markets under consideration, and this ideal behavior is frequently far different from actual observed behavior.

Since stochastic portfolio theory is descriptive, it is applicable under a wide range of assumptions and conditions that may hold in real markets. Unlike current methods used in mathematical finance, stochastic portfolio theory is consistent with either equilibrium or disequilibrium and with either arbitrage or no-arbitrage. It can be applied to the optimization of actual portfolios and to the analysis of real stock markets.

Chapter 1 introduces the basic definitions and preliminary results that are used throughout the rest of the book. The definitions of stocks and portfolios, although technically equivalent to the conventional definitions, differ from a philosophical perspective. The model presented here is based on the growth rate, or logarithmic rate of return, of the stocks and portfolios, and not on the ordinary (arithmetic) rate of return. From this new perspective, features of stock and portfolio behavior that have heretofore been obscure now stand out clearly.

As an elementary result, it is shown that the growth rate of a stock or portfolio determines its long-term behavior, and that the rate of return becomes irrelevant over time. It is then shown that the growth rate of a portfolio depends not only on the growth rates of its component stocks, but also on their variances and covariances. This result differs significantly from the conventional theory, where the portfolio rate of return is simply the weighted average of the component stocks' rates of return. This difference leads to portfolio optimization in which the covariances of the stocks play a greatly increased role. This form of optimization has a practical advantage, since the variances and covariances are much more amenable to statistical analysis than are the rates of return.

The market, consisting of all the stocks, is studied as a portfolio. The *market weight* of a stock is the ratio of the capitalization of that stock to the total capitalization of the market, so all market weights are numbers between zero and one. The *market portfolio* is the portfolio that holds each stock at its market weight.

Chapter 2, "Stock market behavior and diversity," develops minimal conditions for the structural stability of an equity market. Diversity measures how uniformly capital is distributed among the stocks in the market. In a *diverse* market, the market weight of any stock cannot approach one. In actual markets, this condition would certainly be ensured by any credible antitrust law. In a *weakly diverse* market, the time average of the maximum of all the market weights cannot approach one. It is not difficult to see that a diverse market is also weakly diverse, but a weakly diverse market need not be diverse. These diversity conditions are both quite weak, and are obviously consistent with the structure of the U.S equity market, or any other equity market of any significance.

Despite the fact that diversity and weak diversity are not very restrictive, many simple hypothetical markets fail to satisfy either of these conditions. For example, if the stocks in a market have constant growth rates, or if they all have the same growth rate, then the market is not weakly diverse. In fact, these markets are the "opposite" of diverse: the time average of the maximum of the market weights approaches one with probabilistic certainty.

As a measure of diversity, the *entropy* of the market is introduced. Entropy has been used in physics and mathematics as a measure of diversity for some time, and it is natural to introduce it here. However, the interesting point here is that the entropy measure *generates* a portfolio. The log-return on this portfolio, called the *entropy-weighted* portfolio, is related to the market's log-return by a stochastic differential equation. This equation allows probability-one conclusions to be drawn about the relative performance of the entropy-weighted portfolio based on the stability of the entropy of the market. This establishes a connection between market diversity and the relative return of a portfolio. This connection is generalized and expanded in the next chapter.

In Chapter 3, "Functionally generated portfolios," the portfolio construction technique used for the entropy-weighted portfolio is extended to more general functions of the market weights. These functions are called *portfolio generating functions* and the portfolios they generate are called *functionally generated portfolios*. The return on such a portfolio satisfies a stochastic differential equation similar to the equation for the entropy-weighted portfolios.

The stochastic differential equation associated with a functionally generated portfolio decomposes the logarithmic relative return of the portfolio into two components. The first component is the logarithmic change in the value of the generating function, and this term can be controlled by bounds on the value of the generating function. The second term is called the *drift process*, and this is usually an expression involving the relative variances and covariances of the stocks in the portfolio.

The entropy function is a member of a class of functions called *measures of diversity*. Measures of diversity generate portfolios called *diversity-weighted* portfolios, of which the entropy-weighted portfolio is an example. Relative to the market portfolio, diversity-weighted portfolios are overweighted in the smaller stocks, and underweighted in the larger stocks. The drift processes for diversity-weighted portfolios are increasing, so these portfolios outperform the market over periods in which the level of diversity of the market is maintained. Since in a stable market, the market diversity is likely to be maintained, this condition is frequently met in practice. In one example of this chapter, the increasing nature of the drift process for a particular measure of diversity is exploited to show that the no-arbitrage hypothesis of mathematical finance fails to hold in a weakly diverse market. Another measure of diversity defined here, $\mathbf{D}_p$, has been used in practice to construct portfolios for the management of institutional equity portfolios.

In Chapter 4, "Portfolios of stocks selected by rank," the idea of considering stocks by their rank rather than by their name is introduced. The *capital distribution* of the market is defined to be the family of ranked market weights, starting with the largest weight and going to the smallest. The results of the previous chapter are extended here to show that certain functions of the ranked market weights generate portfolios. Again the relative return of a functionally generated portfolio satisfies a stochastic differential equation, but now the relative return has an additional component. The new component depends on the *local times* that measure the frequency of changes in rank among the stocks.

This additional component provides new insight in at least two situations. First, for portfolios composed of stocks from an index, e.g., an index composed of the largest 1000 stocks in the market, the local time component evaluates the effect on the relative return of the portfolio of stocks entering or leaving the index because of rank changes. It also provides a mathematically rigorous explanation of the *size effect*, the tendency of smaller stocks to outperform larger stocks over the long term.

However, the most important application of the idea of ranked market weights is not to portfolio construction, but to provide a basis for a structural model for equity markets which have a stable distribution of capital. This concept of stability is made precise and developed in the next chapter.

Chapter 5, "Stable models for the distribution of capital," presents a structural theory for equity markets in which the capital distribution exhibits stability over time. The *capital distribution curve* refers to the log-log plot of the market weights arranged in descending order, i.e., the logarithms of the market weights versus the logarithms of their respective ranks. If this curve is roughly a straight line, then the capital distribution follows a *Pareto* distribution. Much is known about the Pareto distribution, and attempts have been made to apply it to the capital distribution of the market. However, empirically the capital distribution curve does not resemble a straight line, but instead is notably concave (dome-shaped).

An *asymptotically stable* market is a market in which the frequency of changes in rank among the stocks is stable over the long term. For asymptotically stable markets, it is shown that there are certain *characteristic parameters* that determine the shape of the capital distribution curve. The U.S. equity market is analyzed and appears to be asymptotically stable, and the values of the characteristic parameters for the U.S. market are estimated.

Finally, a *first-order model* is constructed for an asymptotically stable market, and it is shown that the first-order model has (almost) the same capital distribution curve as the market from which it is derived. For the U.S. market, the capital distribution curve for the first-order model is stable, and does not vary as much over time as the actual capital distribution curve of the market.

In Chapter 6, "Performance of functionally generated portfolios," a number of examples of portfolios of stocks in the U.S. equity market are studied. Market entropy and a diversity-weighted portfolio generated by the generating function $\mathbf{D}_p$ are analyzed. Trading turnover is analyzed for a $\mathbf{D}_p$-generated portfolio, and it is shown that in the U.S. market a $\mathbf{D}_p$-generated portfolio will have about 7% turnover a year if the stocks are traded when their actual weights differ from their theoretical portfolio weights by a factor of .1 or greater.

In this chapter, the explanation of the size effect given in Chapter 4 is tested in the U.S. market. It is shown that the "crossovers" of stocks from a large-stock index to a small-stock index generate about 1% a year superior logarithmic return for the small-stock index. The portfolio containing only the largest stock in the market is analyzed, and this portfolio is shown to underperform the market by about .19% a year.

Chapter 7, "Applications of stochastic portfolio theory," provides examples of various types of applications for the methodology. Optimization models are constructed using the first-order model, and these models enjoy the property that all the necessary parameters, i.e., the growth rates and variances, can be easily calculated. Estimation of these parameters is a serious weakness in classical portfolio theory, and with the first-order model, such estimation is unnecessary. Also discussed here is *diversity-weighted indexing*, the application of the $\mathbf{D}_p$ generating function to popular large-stock indices such as the S&P 500 or the Russell 1000.

It turns out that change in diversity is a significant factor in manager performance, and this factor explained more than half of the variation of manager performance relative to the S&P 500 over the period from 1971 to 1998. The relationship between manager performance and change in diversity is discussed here, as well as a method of directly calculating the effect that changes in the capital distribution have on portfolio return.